Continuous decisions by a committee: median versus average mechanisms☆

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Abstract

A group of strategic agents with diverse private information and interdependent preferences has to take a continuous collective decision. I study the design of the decision–making procedure from the viewpoint of a utilitarian social planner. For uniformly distributed information, the implementation of the average report as decision dominates the implementation of the median report when the set of admissible reports is optimally designed. This is true for any number of agents and for any degree of interdependence. The result extends to a general class of distributions when the number of agents is large.

Keywords: collective decision; median mechanism; average mechanism; optimal delegation; interdependent preferences; no monetary transfers

JEL classification: D71, D72, D82

1. Introduction

The quality of decisions often depends on information which is distributed among agents with correlated but not completely aligned interests. Think of an economic policy decision which affects all members of the European Union. Each member is likely to be better informed about how the decision affects its own country than the other members, but it may also care about the other members' information because of spill–over effects.[1] I am interested in mechanisms for making a continuous collective decision in such environments.

I study such decisions from the viewpoint of a utilitarian social planner in the setting introduced by Grüner and Kiel (2004). Each agent is endowed with an independent private signal, the for him individually optimal decision is however a convex combination of his own private information and the private information of the other agents. Differences between the collective decision and the individually optimal decisions cause quadratic losses for the respective

[1]See Subsection 1.2 in Grüner and Kiel (2004) for a more detailed discussion of applications which fall into this framework. Grüner and Kiel discuss central bank decisions, environmental policy decisions, decisions about the provision of a public good and decisions taken within a firm.

agents. This formulation includes private preferences, common preferences and interdependent preferences as special cases.

The social planner's problem to aggregate the agents' information into a collective decision differs from a mere statistical aggregation problem as the primitives of the aggregation procedure, the agents' reports, may be subject to strategic manipulations by the agents. The performance of a decision–making mechanism depends thus on both, its potential to aggregate information and its vulnerability to strategic manipulations. In particular, the susceptibility of the collective decision to manipulations by agents with extreme opinions is often a major concern when it comes to the design of decision–making procedures.

Decisions from a continuous set of alternatives are often taken according to some version of a trimmed mean mechanisms. That is, each agent submits a report, the most extreme reports are discarded, and the average of the remaining reports is implemented as decision. Prominent examples include the determination of interest rates by banks (e.g., Libor and Euribor)[2] and the determination of scores in sports by a panel of judges (e.g., figure skating or ski jumping).[3] Such mechanisms may however differ strongly in the details, the set of admissible reports and the aggregation rule. Possible aggregation rules range from the average aggregation rule where no report is trimmed to the median aggregation rule where all reports but the median report are trimmed. The report space can be quite large (e.g., the set of all real–valued or all positive numbers) or relatively small (e.g., a small finite set of possible scores).

Trimmed mean mechanisms come along with two instruments for restricting the impact of extremists on the collective decision. By trimming the most extreme submitted reports, the impact of reports which are extreme on a *relative scale* is restricted.[4] By removing extreme reports from the report space, the planner can affect how extreme the submitted reports can be on an *absolute scale*. Both instruments trade–off the potential of the collective decision to be responsive on the agents' information with its vulnerability to strategic exaggerations.

I study the median and the average aggregation rule in combination with general report spaces. This allows me to answer the following kind of questions: Is one of the two instruments better suitable for reducing the vulnerability of the collective decision to strategic manipulations? How is the optimal use of the two instruments affected by the degree of interdependence in preferences and the committee size? Because advances in information technologies have made decisions involving a large number of agents significantly easier to implement, it is specifically

[2]The London Interbank Offered Rates (Libor) are set as follows: "Contributed rates are ranked in descending order and the arithmetic mean of only the middle two quartiles is used to formulate the resulting BBA LIBOR calculation for that particular currency and maturity." See `http://www.bbatrent.com/technical-aspects/setting-bbalibor` (last accessed: May 12, 2015). See also MacKenzie (2008). A similar mechanism determines the Euro Interbank Offered Rates (Euribor). See `http://www.emmi-benchmarks.eu/assets/files/Euribor_tech_features.pdf` (last accessed: May 12, 2015).

[3]The performance of athletes is in many sports measured by a combination of objective and subjective scores. The subjective score is typically determined by a panel of judges according to some trimmed mean mechanism.

[4]MacKenzie (2008) observes that this instrument is used for reducing the vulnerability of the Libor to strategic manipulations. Mollenkamp and Whitehouse (2008) report on banks reporting interest rates strategically.

interesting how the two instruments are optimally used for large committees.

My problem blends issues from two literatures. The collective decision–making literature focuses on the information aggregation properties of specific aggregation rules. The report space is typically assumed to coincide with the set of possible decisions, its optimal design is not an issue (e.g., Grüner and Kiel (2004)). On the other hand, the literature on optimal delegation studies the generally optimal way to utilize the information of a single agent. The "aggregation of information" is then trivial and the design problem reduces to the design of a "report space" (e.g., Alonso and Matouschek (2008)).

Understanding how the aggregation rule interacts with the report space is also of practical importance. Although the designer of the decision–making process may want to stick to a simple and transparent aggregation rule like the median aggregation rule, the average aggregation rule, or a rule with an intermediate amount of trimming, there is often no good reason which prevents her from determining among which reports the agents can choose.

My two main results concern the relative performance of the two aggregation rules when the report space is optimally designed. For the case with uniformly distributed signals which is much discussed in the information transmission and in the delegation literature, I find that the average aggregation rule outperforms the median aggregation rule for any degree of interdependence and for any number of agents (Proposition 5). When the number of agents is large, the result extends to a general class of distributions (Proposition 6). My main results suggest that it is better to restrict the impact of agents which are extreme on an absolute scale than to restrict the impact of the relatively most extreme agents. This is, in particular, true for large committees even though it involves having to deal with a very severe exaggeration problem. It is then near optimal to extract only binary information from each agent (e.g., "left" or "right").

2. Literature

The early literature on collective decision–making by strategic agents focuses on binary decisions and specific mechanisms. Feddersen and Pesendorfer (1996, 1997, 1999) study majority voting with and without abstention in an electoral context. Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1998) study voting under simple majority and supermajority rules in a jury context. Equilibrium behavior conditions on pivotality and information aggregation depends non–trivially on the mechanism choice. Li et al. (2001) show that equilibrium behavior features exaggeration under different voting and scoring mechanisms when the committee members have continuous private information. Schmitz and Tröger (2011) study optimal dominant strategy and optimal Bayesian implementation in a somewhat different setting with two alternatives and continuous information. The only instrument is the probability with which the alternatives are selected. As the agents' utility is linear in the instrument, optimal implementation with a binary decision is specific. In particular, the collective decision cannot depend in a continuous way on the agents' information. Majority rules arise naturally in such a setting.

When the collective decision is non–binary, a planner has instruments which affect the agents' utility non–linearly. This enables the implementation of decisions which depend in a continuous way on the agents' information. There exists some literature on optimal implementation in the case with two agents and private preferences. Martimort and Semenov (2008) derive the optimal deterministic dominant strategy mechanism for the uniform–quadratic case. Dominant strategy implementation is however much more restrictive than Bayesian implementation in which I am interested in. In particular, collective decisions cannot arise as compromises between the agents. Carrasco and Fuchs (2009) derive the optimal stochastic mechanism which is Bayesian incentive compatible for quadratic preferences. They guess the solution and show with Lagrangian methods that it is optimal. Their analysis provides no insights into the case with more than two agents.[5]

To the best of my knowledge, there are no results on optimal Bayesian implementation for the problem with more than two agents.[6] Grüner and Kiel (2004) study Bayesian Nash equilibria of the unrestricted average mechanism and the unrestricted median mechanism in the same setting with more than two agents and interdependent preferences which I consider. They find that the unrestricted average mechanism is better for common preferences, whereas the unrestricted median mechanism is better for private preferences.[7,8]

The literature on optimal delegation considers the problem of an uninformed principal who wants to utilize an agent's information, but who is unable to set monetary incentives. An optimal mechanism corresponds to an optimal delegation set from which the agent can freely pick a decision. Alonso and Matouschek (2008) provide a full characterization of the optimal delegation set for the general quadratic case.[9]

For a setting which corresponds basically to my setting with private preferences, Kawamura (2011) studies the complementary problem in which the planner cannot commit to a mechanism. The agents send cheap talk messages and the planner takes a decision after observing these messages. As in my article, the agents have an incentive to exaggerate their private information. Binary communication emerges endogenously for large numbers of agents.[10]

[5]Börgers and Postl (2009) consider also optimal Bayesian implementation in a setting with two agents and three alternatives, but their setting has a different flavor. The ordinal preferences of the two agents are common knowledge and diametrically opposed. The asymmetry of information concerns only cardinal utility.

[6]For a setting in which a large number of collective decisions has to be taken, Jackson and Sonnenschein (2007) show that first–best decisions can be approximated when decisions can be linked.

[7]See Moldovanu and Shi (2013) and Yildirim (2012) for other collective decision–making problems with interdependent preferences.

[8]Holmström (1984) introduces a subclass of decentralization problems with multiple agents where no information gets coordinated in the decision process (but he analyzes only the case with a single agent). He interprets this as a multi–agent extension to delegation. The average aggregation rule with the optimal report space in my setting can be interpreted as optimal delegation in the sense of Holmström (1984). The average aggregation rule in combination with a given non–optimal report space is studied in a number of articles (e.g., Rausser et al. (2015), De Sinopoli and Iannantuoni (2007), Renault and Trannoy (2005) and Cai (2009)).

[9]Holmström (1984) poses the delegation problem and discusses optimal interval delegation in some examples. Melumad and Shibano (1991) provide a full characterization for the uniform–quadratic case. Martimort and Semenov (2006) give a sufficient condition for interval delegation to be optimal.

[10]See Alonso et al. (2008) for a cheap talk problem with two agents.

3. The model

A committee consisting of $N \in \mathcal{N} := \{N' \in \mathbb{N} | N' \geq 3,\ N' \text{ odd}\}$ agents has to take a collective decision $y \in \mathbb{R}$. I index the agents by $1, \ldots, N$ and I denote generic agents by i and j. Each agent i is privately informed about a real–valued attribute $x_i \in \mathcal{X} := [\underline{t}, \overline{t}]$, his payoff depends, however, also on the other agents' information $x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$. It is $u(y - \theta(x_i, x_{-i})) := -(y - \theta(x_i, x_{-i}))^2$ with $\theta(x_i, x_{-i}) := (1 - \alpha)x_i + \alpha/(N - 1) \cdot \sum_{j \neq i} x_j$ and $\alpha \in [0, (N - 1)/N]$. That is, each agent i suffers a quadratic loss when the collective decision y differs from his individually optimal decision $\theta(x_i, x_{-i})$. The parameter α describes the degree of alignment between the agents' preferences. This formulation includes private preferences ($\alpha = 0$), common preferences ($\alpha = (N - 1)/N$) and interdependent preferences ($\alpha \in (0, (N - 1)/N)$) as special cases.

The attribute x_i is the realization of a $\mathcal{X}$–valued random variable X_i. Let $X := (X_1, \ldots, X_N)$, $X_{-i} := (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_N)$ and $\overline{X} := (\sum_i X_i)/N$. I use lower case letters to denote realizations of these random variables. $X_1, \ldots, X_N$ are independently drawn from a common distribution function F. Let $\mu := \mathbf{E}_{X_i}[X_i]$, $\sigma^2 := \mathrm{Var}_{X_i}[X_i]$ and $\eta := F^{-1}(1/2)$. I assume that F is twice continuously differentiable with a strictly positive density function f and that F is normalized such that $\mu = 0$. Let $\mathcal{F}$ be the set of all such distribution functions. For some of my results I will need to impose additional structure. Let $\mathcal{F}_h := \{F \in \mathcal{F} | f/(1 - F)$ is strictly increasing and f/F is strictly decreasing$\}$ and $\mathcal{F}_s := \{F \in \mathcal{F} | \eta = 0\}$. $F \in \mathcal{F}_h$ is a monotonicity assumption on the hazard rate $f/(1 - F)$ and the reversed hazard rate f/F. $F \in \mathcal{F}_s$ can be interpreted as a minimal symmetry assumption as it requires that the median of the distribution η coincides with its (normalized) mean $\mu = 0$.

The collective decision is taken according to a decentralized decision–making mechanism (ϕ, R) with $R \in \mathcal{R} := \{R \subset \mathbb{R} | R \text{ compact and non–empty}\}$ and $\phi : R^N \to \mathbb{R}$. R describes the common report space and ϕ describes the rule according to which individual reports are aggregated into a collective decision.[11] The timing is as follows: First, each agent i learns his attribute x_i as a private signal. Second, all agents simultaneously submit reports $r_i \in R$. Third, the collective decision $y = \phi(r_1, \ldots, r_n)$ is taken and payoffs realize. By considering this timing, I implicitly assume that participation in the mechanism is mandatory[12] and that there

[11]Restricting attention to compact report spaces is without loss of generality. See the reasoning in Footnote 15 of Alonso and Matouschek (2008).

[12]Mandatory participation is standard in the literature on collective decision–making. If participation is voluntary, equilibrium behavior can feature non–participation with a positive probability. An example can be constructed which is reminiscent of Feddersen and Pesendorfer (1996) and Feddersen and Pesendorfer (1999) who analyze majority voting with abstention in a setting with two alternatives. If the report space is binary and the decision is the median report, the decision–making mechanism corresponds to majority voting with two alternatives. If preferences are not private, an agent who is indifferent between the two reports after learning his attribute may strictly prefer not to participate. However, even if the social planner cannot force the agents to participate, she can augment any mechanism by a default report which is used for aggregation in case of non–participation. There exists then an equilibrium in which each agent participates and reports as in the here analyzed game with mandatory participation.

is no informative communication before reporting.[13]

I am interested in general report spaces $R \in \mathcal{R}$ in combination with the two aggregation rules where the decision is either the average report $\phi_a(r_1, \dots, r_N) := 1/N \cdot \sum_i r_i$ or the median report $\phi_m(r_1, \dots, r_N) := \mathrm{m}(r_1, \dots, r_N)$. To explain my results, I will also discuss binary report spaces ($R \in \mathcal{R}_B := \{\{\underline{r}, \overline{r}\} \subset \mathbb{R} | \underline{r} \leq \overline{r}\}$), interval report spaces ($R \in \mathcal{R}_I := \{[\underline{r}, \overline{r}] \subset \mathbb{R} | \underline{r} \leq \overline{r}\}$) and unrestricted reporting ($R = \mathbb{R}$).

I denote the reporting strategy of agent i by $\widehat{r}_i : \mathcal{X} \to R$. Let $\widehat{r} := (\widehat{r}_1, \dots, \widehat{r}_N)$ and let $\widehat{r}_{-i} := (\widehat{r}_1, \dots, \widehat{r}_{i-1}, \widehat{r}_{i+1}, \dots, \widehat{r}_N)$. Agent i's interim expected payoff from report r_i when his attribute is x_i and when the other agents report according to $\widehat{r}_{-i}$ is $U_i(x_i, r_i, \widehat{r}_{-i}) := \mathbf{E}_X[u(\phi(r_i, \widehat{r}_{-i}(X_{-i})) - \theta(X_i, X_{-i}))|X_i = x_i]$. $\widehat{r}$ constitutes a Bayesian Nash equilibrium (BNE) of the game implied by the mechanism (ϕ, R) if for all i and for all $x_i \in \mathcal{X}$, $\widehat{r}_i(x_i) \in \arg\max_{r_i \in R} U_i(x_i, r_i, \widehat{r}_{-i})$. As the setting is symmetric, I am interested in symmetric BNE (sBNE). $\widehat{r}_i$ constitutes a sBNE if $(\widehat{r}_i, \dots, \widehat{r}_i)$ constitutes a BNE. I call any function $\widehat{y} : \mathcal{X}^N \to \mathbb{R}$ a decision function. The decision function $\widehat{y}$ is implemented by the mechanism (ϕ, R) and the sBNE $\widehat{r}_i$ if $\widehat{y} = \phi \circ (\widehat{r}_i, \dots, \widehat{r}_i)$.

The decision–making mechanism $(R, \phi) \in \mathcal{R} \times \{\phi_a, \phi_m\}$ is designed by a utilitarian social planner who knows the distribution of X but not its realization x. As is standard in the design literature, I assume that the planner selects also which sBNE of the game implied by her mechanism choice is played. Mechanism choice and equilibrium selection together determine the decision function $\widehat{y}$ which is implemented and therewith the planner's expected utility $U_0(\widehat{y}) := \mathbf{E}_X[\sum_i u(\widehat{y}(X) - \theta(X_i, X_{-i}))]$. As $U_0(\widehat{y}) = V(\widehat{y}) + \overline{U}_0$ with $V(\widehat{y}) := N \cdot \mathbf{E}_X[u(\widehat{y}(X) - \overline{X})]$ and $\overline{U}_0 := N \cdot \mathbf{E}_X[u(\overline{X} - \theta(X_i, X_{-i}))]$, the decision function affects $U_0(\widehat{y})$ only through $V(\widehat{y})$.[14] I can thus employ $V(\widehat{y})$ without loss of generality as the welfare functional which the planner strives to maximize. The structure of $V(\widehat{y})$ gives rise to the interpretation that the planner's preferences are like those of an uninformed agent who is not biased into the direction of any particular attribute.

3.1. Centralized decision–making under different informational assumptions

As a benchmark case, I consider optimal centralized decision–making under different informational assumptions. This endows me with upper bounds on welfare when information is controlled by strategic agents and the collective decision reflects a certain kind of information. Suppose the planner directly chooses a decision $y \in \mathbb{R}$ after learning the realization of some statistic $\mathrm{Stat}(X)$. Due to the quadratic nature of the welfare functional, the optimal decision corresponds to the expectation of the socially optimal decision conditional on the available information, $\widehat{y}^{*|\mathrm{Stat}(X)}(x) := \mathbf{E}_X[\overline{X}|\mathrm{Stat}(X) = \mathrm{Stat}(x)]$. The subsequent lemma states a formula

[13] In the augmented game in which the agents can communicate before reporting, there generally exists a babbling equilibrium in which communication is uninformative. The analysis in this article applies to this class of equilibria. The importance of such equilibria is supported by a literature in social psychology which finds that only very little diverse information is shared when a group discusses before it takes a collective decision. See Strasser and Titus (1985), Strasser and Titus (1987) and Strasser et al. (1989).

[14] See Appendix B for a proof of the decomposition of $U_0(\widehat{y})$.

for the attained welfare $V^{*|\mathrm{Stat}(X)} := V(\widehat{y}^{*|\mathrm{Stat}(X)})$.

Lemma 1 *Let* $\mathrm{Stat}(X)$ *be any statistic of* X. *(a)* $V^{*|\mathrm{Stat}(X)} = -\sigma^2 + N \cdot \mathbf{E}_X[\mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)]^2]$. *(b) If there exists a function* S *such that* $\mathrm{Stat}(X) = (\mathrm{S}(X_1), \ldots, \mathrm{S}(X_N))$, *then* $V^{*|\mathrm{Stat}(X)} = -\sigma^2 + \mathbf{E}_{X_i}[\mathbf{E}_{X_i}[X_i|\mathrm{S}(X_i)]^2]$.

The upper bounds on welfare simplify when I impose additional structure on the available information $\mathrm{Stat}(X)$. When the planner is fully informed ($\mathrm{Stat}(X) = X$), informed about whether each attribute exceeds a threshold t ($\mathrm{Stat}(X) = (\mathrm{sgn}(X_1 - t), \ldots, \mathrm{sgn}(X_N - t))$) or uninformed ($\mathrm{Stat}(X) = 0$), I obtain an upper bound which holds under general conditions.[15] When she is informed about the median attribute ($\mathrm{Stat}(X) = \mathrm{m}(X)$) or about the sign of the median attribute ($\mathrm{Stat}(X) = \mathrm{sgn}(\mathrm{m}(X))$), I obtain an asymptotic upper bound for distributions satisfying my minimal symmetry assumption. If the attributes are uniformly distributed, I obtain also for $\mathrm{Stat}(X) = \mathrm{m}(X)$ a non–asymptotic upper bound.

Lemma 2 *Let* $t \in \mathcal{X}$ *and define* $c(t) := F(t)\mathbf{E}_{X_i}[X_i|X_i \leq t]^2 + (1 - F(t))\mathbf{E}_{X_i}[X_i|X_i \geq t]^2$. *(a)* $V^{*|X} = 0$. *(b)* $V^{*|(\mathrm{sgn}(X_1-t),\ldots,\mathrm{sgn}(X_N-t))} = -\sigma^2 + c(t)$. *(c)* $V^{*|0} = -\sigma^2$. *(d) If* $F \in \mathcal{F}_s$, *then* $\lim_{N\to\infty} V^{*|\mathrm{m}(X)} = V^{*|(\mathrm{sgn}(X_1),\ldots,\mathrm{sgn}(X_N))}$. *(e) If* $F \in \mathcal{F}_s$, *then* $\lim_{N\to\infty} V^{*|\mathrm{sgn}(\mathrm{m}(X))} = V^{*|(\mathrm{sgn}(X_1),\ldots,\mathrm{sgn}(X_N))} - (1 - 2/\pi)c(0)$. *(f) If* $X_i \sim \mathrm{U}[-\overline{t}, \overline{t}]$, $V^{*|\mathrm{m}(X)} = -\sigma^2 + 1/4 \cdot (N+1)^2/((N+1)^2 - 1) \cdot \overline{t}^2$.

None of the bounds depends on the degree of alignment in preferences α. The alignment in preferences affects the agents' incentives and therewith how information can be reflected in the collective decision when it is controlled by strategic agents, but it does not affect the welfare maximizing use of exogenously given information.

4. Strategic reporting behavior

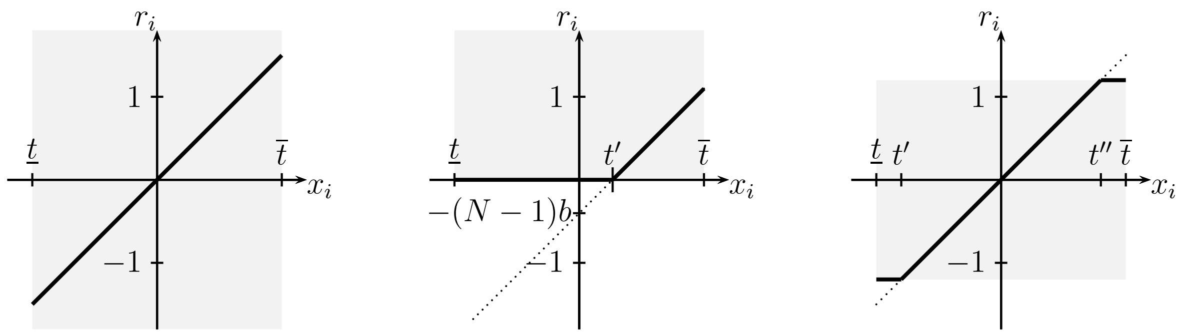
How do strategic agents report? For any mechanism $(\phi, R) \in \{\phi_a, \phi_m\} \times \mathcal{R}$, I can answer this question in two steps. I will at first consider how an agent i with attribute x_i does optimally respond to a reporting behavior $\widehat{r}_{-i} = (\widehat{r}_j, \ldots, \widehat{r}_j)$ by the other agents. I will call the report $r_i \in \mathbb{R}$ which he would choose in the hypothetical situation in which he is not constrained in his report choice as his *preferred report.* Having understood this, I can derive how agent i does *actually report* in equilibrium when his report choice is constrained by the report space.

4.1. Strategic reporting under the average aggregation rule

Consider $\phi = \phi_a$. Agent i suffers a quadratic loss from the distance between the collective decision $\phi_a(r_i, \widehat{r}_{-i}(X_{-i}))$ and the for him individually optimal decision $\theta(x_i, X_{-i})$. Because of the quadratic nature of his loss function, his expected payoff is completely determined by the expected distance and the variance of the distance conditional on his private information:

$$\begin{aligned} U_i(x_i, r_i, \widehat{r}_{-i}) &= -\mathbf{E}_X[\phi_a(r_i, \widehat{r}_{-i}(X_{-i})) - \theta(X_i, X_{-i})|X_i = x_i]^2 \\ &\quad -\mathrm{Var}_X[\phi_a(r_i, \widehat{r}_{-i}(X_{-i})) - \theta(X_i, X_{-i})|X_i = x_i]. \end{aligned}$$

[15] I use a binary version of the signum function where $\mathrm{sgn}(x_i) = -1$ if $x_i < 1$ and $\mathrm{sgn}(x_i) = 1$ if $x_i \geq 1$.



(a) $R = \mathbb{R}$ (b) $R = \mathbb{R}_+$ (c) $R = [-6/5, 6/5]$

Figure 1: Reporting under the average aggregation rule [$N = 3$, $\alpha = 1/2$, $X_i \sim \mathrm{U}[-1, 1]$]

As the collective decision is under the average aggregation rule additively separable in his own report and the other agents' reports, agent i's report shifts the expected decision without affecting the variance expression. It follows that he chooses his report to shift the expected decision $\mathbf{E}_X[\phi_a(r_i, \widehat{r}_{-i}(X_{-i}))|X_i = x_i] = 1/N \cdot (r_i + (N-1)\mathbf{E}_{X_j}[\widehat{r}_j(X_j)])$ as close as possible to his expected individually optimal decision $\mathbf{E}_X[\theta(X_i, X_{-i})|X_i = x_i] = (1-\alpha)x_i$. His preferred report is thus

$$\widehat{r}_i^a(x_i, b) := N(1-\alpha)x_i - (N-1)b \tag{1}$$

with $b = \mathbf{E}_{X_j}[\widehat{r}_j(X_j)]$. It countervails the expected bias b of each of the $N-1$ other agents' reporting behavior and it takes the averaging by the aggregation rule into account by exaggerating his expected individually optimal decision $(1-\alpha)x_i$ by the factor N.

What does this imply for the equilibrium reporting behavior? Suppose first reporting is unrestricted ($R = \mathbb{R}$). Agent i's preferred report is then always admissible. Moreover, as $\mathbf{E}_{X_i}[N(1-\alpha)X_i] = 0$, no agent's preferred reporting behavior is biased given that the other agents' reporting behavior is not biased. $\widehat{r}_i^{a,\mathbb{R}}(x_i) := N(1-\alpha)x_i$ constitutes the unique sBNE (see Proposition 2 in Grüner and Kiel (2004)). See Figure 1a for an illustration. If reporting is restricted, two differences may occur: First, a non–zero bias b may prevail in a sBNE. Although each agent's incentive to offset the biases of the other agents serves as a force that works against a non–zero bias in a symmetric equilibrium, a complete offset may not be possible. For example, a strictly positive bias prevails when only positive reports are admissible. Second, an agent's preferred report may not be admissible. He chooses then the closest admissible report. See Figure 1b and Figure 1c for an illustration of the case with a strictly positive bias and a zero bias. The gray area indicates which reports are admissible; the dotted curve depicts the preferred reporting behavior; the solid curve depicts the actual reporting behavior.

Proposition 1 *Consider the game implied by any mechanism (ϕ_a, R). (a) A sBNE exists. (b) $\widehat{r}_i$ constitutes a sBNE if and only if $\widehat{r}_i$ selects for any x_i an admissible report which lies closest to $\widehat{r}_i^a(x_i, b)$ with $b = \mathbf{E}_{X_i}[\widehat{r}_i(X_i)]$. (c) Any sBNE possesses the same bias b and attains the same welfare.*

Proposition 1 (b) describes the equilibrium reporting behavior only implicitly. The subsequent two corollaries describe it explicitly for the two important special cases in which the report space is either an interval or binary.

Corollary 1 *Define* $R^{a,I}(t',t'') := [\widehat{r}_i^a(t', b^{a,I}(t',t'')), \widehat{r}_i^a(t'', b^{a,I}(t',t''))]$ *with* $b^{a,I}(t',t'') := (1-\alpha)(F(t')t' + \int_{t'}^{t''} x_i f(x_i)\mathrm{d}x_i + (1-F(t''))t'')$. *(a) Consider the game implied by any mechanism* $(\phi_a, R^{a,I}(t',t''))$ *with* $t', t'' \in \mathcal{X}$ *and* $t' \leq t''$. *Then,* $\widehat{r}_i$ *defined by* $\widehat{r}_i(x_i) = \widehat{r}_i^a(t', b^{a,I}(t',t''))$ *if* $x_i < t'$, $\widehat{r}_i(x_i) = \widehat{r}_i^a(x_i, b^{a,I}(t',t''))$ *if* $x_i \in [t', t'']$ *and* $\widehat{r}_i(x_i) = \widehat{r}_i^a(t'', b^{a,I}(t',t''))$ *if* $x_i > t''$ *constitutes the unique sBNE. (b) If* $\widehat{r}_i$ *constitutes a non–constant sBNE of the game implied by some mechanism* (ϕ_a, R) *with* $R \in \mathcal{R}_I$, *then there exist* $t', t'' \in \mathcal{X}$ *with* $t' < t''$ *such that* $\widehat{r}_i$ *constitutes a sBNE of the game implied by the mechanism* $(\phi_a, R^{a,I}(t',t''))$.

Corollary 2 *Define* $R^{a,B}(t,\delta) := \{\underline{r}^{a,B}(t,\delta), \overline{r}^{a,B}(t,\delta)\}$ *with* $\overline{r}^{a,B}(t,\delta) := (1-\alpha)t+\delta$, $\underline{r}^{a,B}(t,\delta) := (1-\alpha)t - k(t)\delta$ *and* $k(t) := ((N-1)(1-F(t)) + 1/2)/((N-1)F(t) + 1/2)$. *(a) Consider the game implied by any mechanism* $(\phi_a, R^{a,B}(t,\delta))$ *with* $t \in \mathcal{X}$ *and* $\delta \geq 0$. *In any sBNE* $\widehat{r}_i$, $\widehat{r}_i(x_i) = \underline{r}^{a,B}(t,\delta)$ *if* $x_i < t$ *and* $\widehat{r}_i(x_i) = \overline{r}^{a,B}(t,\delta)$ *if* $x_i > t$. *(b) If* $\widehat{r}_i$ *constitutes a non–constant sBNE of the game implied by some mechanism* (ϕ_a, R) *with* $R \in \mathcal{R}_B$, *then there exist* $t \in \mathcal{X}$ *and* $\delta > 0$ *such that* $\widehat{r}_i$ *constitutes a sBNE of the game implied by the mechanism* $(\phi_a, R^{a,B}(t,\delta))$.

Because a sBNE exists for any average mechanism (ϕ_a, R) by Proposition 1 (a) and because any sBNE attains the same welfare by Proposition 1 (c), it is meaningful to refer to equilibrium welfare without referring to an equilibrium. This allows me to use henceforth the notation $V(\phi_a, R)$ for the equilibrium welfare of the average mechanism (ϕ_a, R).

4.2. Strategic reporting under the median aggregation rule

Consider now $\phi = \phi_m$. To discuss the strategic reporting behavior, it is useful to introduce an equilibrium refinement which helps me to classify the multitude of existing sBNE. Define $R^*(\widehat{r}_j) := \{r_i \in R | \mathrm{Prob}_X[\mathrm{m}(\widehat{r}_j(X_1), \dots, \widehat{r}_j(X_{i-1}), r_i, \widehat{r}_j(X_{i+1}), \dots, \widehat{r}_j(X_N)) = r_i] > 0\}$. $R^*(\widehat{r}_j)$ is the set containing all admissible reports of agent i which would be selected with positive probability as the collective decision by the median aggregation rule when all other agents report according to $\widehat{r}_j$. I say $\widehat{r}_i$ constitutes a sBNE* if $\widehat{r}_i$ constitutes a sBNE and $\mathrm{cl}(\widehat{r}_i(\mathcal{X})) = \mathrm{cl}(R^*(\widehat{r}_i))$. An example for a strategy which violates the additional property is $\widehat{r}_i(x_i) = x_i$ if $x_i \in [\underline{t}, \overline{t})$ and $\widehat{r}_i(\overline{t}) = 2\overline{t}$. I have then $\mathrm{cl}(\widehat{r}_i(\mathcal{X})) = \mathcal{X} \cup \{2\overline{t}\}$ but $\mathrm{cl}(R^*(\widehat{r}_i)) = \mathcal{X}$.[16]

Crucial for agent i's reporting incentives is that his report has only a marginal effect on the collective decision when it happens to be the median report. By learning that this is the case, he can infer something about the other agents' attributes. His preferred report corresponds

[16]sBNE can exhibit "unreasonable" behavior which cannot arise in sBNE*. As an example, consider the mechanism $(\phi_m, \{-10^6, 0, 10^6\})$ in a setting with $\mathcal{X} = [-1, 1]$. It is then common knowledge that each individually optimal decision lies in $[-1, 1]$. The reporting behavior defined by $\widehat{r}_i(x_i) = 0$ for $x_i \in [-1, 0) \cup (0, 1)$, $\widehat{r}_i(0) = -10^6$ and $\widehat{r}_i(1) = 10^6$ constitutes nevertheless a sBNE. Although each agent's individually optimal decision is monotonic in his attribute, his reporting behavior is non–monotonic. Moreover, he chooses for some attributes a report which lies far away from any possibly individually optimal decision. Both properties rely on reports which are in equilibrium selected as decision with probability zero and which do not lie close to reports which are selected with a positive probability.

to his expected individually optimal decision conditional on what he can infer from his report being selected as collective decision.[17] This gives me an implicit characterization of agent i's preferred report: $r_i = \mathbf{E}_X[\theta(X_i, X_{-i})|\mathrm{m}(r_i, \widehat{r}_{-i}(X_{-i})) = r_i, X_i = x_i]$. By imposing additional structure on what the agent infers, I can obtain an explicit characterization. In any sBNE* in strictly increasing strategies, an agent can infer that he has the median attribute when his report is selected as decision. Conditional on this inference, his preferred report is explicitly given by

$$\begin{aligned}\widehat{r}_i^m(x_i) &:= \mathbf{E}_X[\theta(X_i, X_{-i})|\mathrm{m}(X_i, X_{-i}) = X_i, X_i = x_i] \\ &= (1-\alpha)x_i + \alpha h(x_i) \qquad (2)\end{aligned}$$

with $h(x_i) := (\mathbf{E}_{X_j}[X_j|X_j \leq x_i] + \mathbf{E}_{X_j}[X_j|X_j \geq x_i])/2$. It will turn out that the function $\widehat{r}_i^m(x_i)$ is also crucial for the characterization of any sBNE* which is not strictly increasing.

How does the actual equilibrium reporting behavior look like? Suppose first reporting is unrestricted ($R = \mathbb{R}$). $\widehat{r}_i^m(x_i)$ is then always admissible. Strict monotonicity of $\widehat{r}_i^m$ renders $\widehat{r}_i^{m,\mathbb{R}} := \widehat{r}_i^m$ a sBNE*. $\widehat{r}_i^{m,\mathbb{R}}$ constitutes the unique strictly increasing sBNE* (see Proposition 1 in Grüner and Kiel (2004)). If reporting is restricted, the equilibrium characterization becomes more involved. An agent's preferred report may then not be admissible. As this induces pooling of different attributes on the same report, equilibrium strategies may only be weakly monotonic. This complicates the derivation of which report an agent prefers conditional on submitting the median report. The characterization of the entire set of sBNE* looks nevertheless fairly simple: If $\widehat{r}_i$ selects for any x_i a report from R which lies closest to $\widehat{r}_i^m(x_i)$, $\widehat{r}_i$ constitutes a sBNE*. Any other sBNE* differs only by the virtual exclusion of extreme reports above an upper bound r'' and below a lower bound r' from the report space. Any $\widehat{r}_i$ which selects for any x_i the closest report from a virtually narrower report space $R_\subseteq \in \mathcal{R}_\subseteq(R) := \{R' \subseteq R | \exists r', r'' \in R : R' = R \cap [r', r''], R' \neq \emptyset\}$ also constitutes a sBNE*. This is because if no agent $j \neq i$ chooses reports outside $[r', r'']$, then such reports can never become the collective decision. This renders it also for agent i weakly optimal not to choose such reports. The following proposition characterizes all sBNE* and it shows that the restriction to sBNE* is without loss of generality from a welfare perspective.

Proposition 2 *Consider the game implied by any mechanism (ϕ_m, R). (a) For any sBNE there exists a sBNE* which attains the same welfare. (b) A sBNE* exists. (c) $\widehat{r}_i$ constitutes a sBNE* if and only if there exists $R_\subseteq \in \mathcal{R}_\subseteq(R)$ such that $\widehat{r}_i$ selects for any x_i a report from $R_\subseteq$ which lies closest to $\widehat{r}_i^m(x_i)$ and $\mathrm{cl}(\widehat{r}_i(\mathcal{X})) = \mathrm{cl}(R^*(\widehat{r}_i))$. (d) Any sBNE* with the same $R_\subseteq$ attains the same welfare.*

sBNE* are not unique for any non–trivial report space. They differ in the responsiveness of the implemented decision function on the median attribute and can be ordered partially. A

[17] A similar effect is observed for majority voting in settings with two alternatives, private information and common values. See Feddersen and Pesendorfer (1996) for the effect in an electoral context (the swing voter's curse) and Austen-Smith and Banks (1996) for the effect in a jury context.

maximum responsive sBNE* exists and is described by $R_{\subseteq} = R$. The subsequent two corollaries describe the maximum responsive sBNE* for the two important special cases in which the report space is either an interval or binary.

Corollary 3 *Define $R^{m,I}(t',t'') := [\widehat{r}_i^m(t'), \widehat{r}_i^m(t'')]$. (a) Consider the game implied by any mechanism $(\phi_m, R^{m,I}(t',t''))$ with $t', t'' \in \mathcal{X}$ and $t' \leq t''$. Then, $\widehat{r}_i$ defined by $\widehat{r}_i(x_i) = \widehat{r}_i^m(t')$ if $x_i < t'$, $\widehat{r}_i(x_i) = \widehat{r}_i^m(x_i)$ if $x_i \in [t', t'']$ and $\widehat{r}_i(x_i) = \widehat{r}_i^m(t'')$ if $x_i > t''$ constitutes the unique maximum responsive sBNE*. (b) If $\widehat{r}_i$ constitutes a non–constant sBNE* of the game implied by some mechanism (ϕ_m, R) with $R \in \mathcal{R}_I$, then there exist $t', t'' \in \mathcal{X}$ with $t' < t''$ such that $\widehat{r}_i$ constitutes a maximum responsive sBNE* of the game implied by the mechanism $(\phi_m, R^{m,I}(t',t''))$.*

Corollary 4 *Define $R^{m,B}(t,\delta) := \{\underline{r}^{m,B}(t,\delta), \overline{r}^{m,B}(t,\delta)\}$ with $\overline{r}^{m,B}(t,\delta) := \widehat{r}_i^m(t) + \delta$ and $\underline{r}^{m,B}(t,\delta) := \widehat{r}_i^m(t) - \delta$. (a) Consider the game implied by any mechanism $(\phi_m, R^{m,B}(t,\delta))$ with $t \in \mathcal{X}$ and $\delta \geq 0$. In any maximum responsive sBNE* $\widehat{r}_i$, $\widehat{r}_i(x_i) = \underline{r}^{m,B}(t,\delta)$ if $x_i < t$ and $\overline{r}^{m,B}(t,\delta)$ if $x_i > t$. (b) If $\widehat{r}_i$ constitutes a non–constant sBNE* of the game implied by some mechanism (ϕ_m, R) with $R \in \mathcal{R}_B$, then there exist $t \in \mathcal{X}$ and $\delta > 0$ such that $\widehat{r}_i$ constitutes a maximum responsive sBNE* of the game implied by the mechanism $(\phi_m, R^{m,B}(t,\delta))$.*

Restricting attention to sBNE* is by Proposition 2 (a) without loss of generality from a design perspective. As any non–maximum responsive sBNE* of a mechanism (ϕ_m, R) is the maximum responsive sBNE* of a mechanism $(\phi_m, R_{\subseteq})$ with a smaller report space, it is further without loss of generality to restrict attention to maximum responsiveness. Because a maximum responsive sBNE* exists by Proposition 2 (b) and (c) and because any maximum responsive sBNE* attains the same welfare by Proposition 2 (d), it is meaningful to refer to welfare without referring to an equilibrium. This allows me to use henceforth the notation $V(\phi_m, R)$ for the equilibrium welfare of the median mechanism (ϕ_m, R).

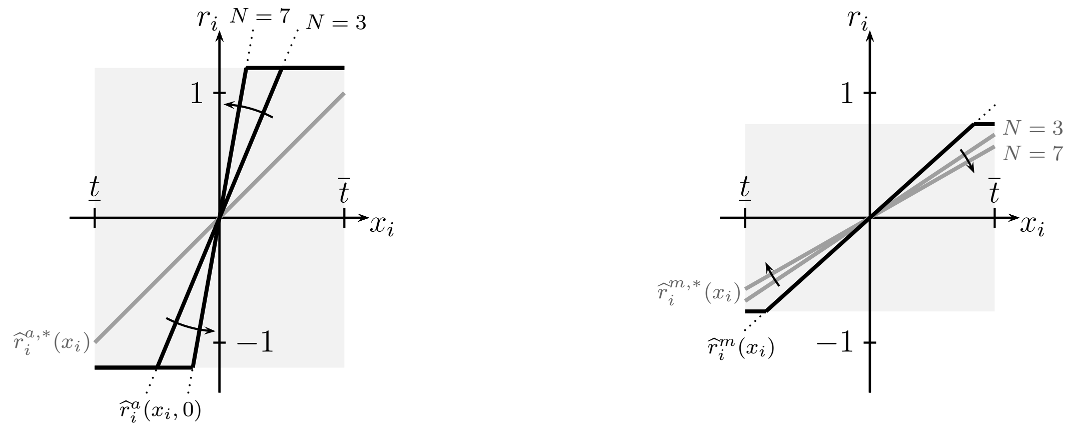
5. The design problem

5.1. Incentives for strategic exaggeration

How does an agent's objective differ from that of the social planner? Unless preferences are completely aligned, the for agent i individually optimal decision can be written as $\theta(x_i, x_{-i}) = \beta\overline{x} + (1-\beta)x_i$ with $\beta < 1$. That is, the individually optimal decision of agent i is biased into the direction of his own attribute x_i relative to the socially optimal decision $\overline{x}$. This gives him an incentive to manipulate the collective decision strategically. How this manipulation incentive expresses itself differs for mechanisms relying on different aggregation rules. To understand how the agent tries to manipulate the collective decision, I need to compare his preferred reporting behavior with how the social planner would like him to report. Let $\widehat{r}_i^{a,*}$ (resp. $\widehat{r}_i^{m,*}$) be the weakly increasing reporting strategy $\widehat{r}_i : \mathcal{X} \to \mathbb{R}$ which maximizes $V(\phi \circ (\widehat{r}_i, \ldots, \widehat{r}_i))$ with $\phi = \phi_a$ (resp. $\phi = \phi_m$). $\widehat{r}_i^{a,*}$ (resp. $\widehat{r}_i^{m,*}$) describes the socially optimal reporting behavior under the aggregation rule $\phi = \phi_a$ (resp. $\phi = \phi_m$).

Consider $\phi = \phi_a$. The first–best decision $\widehat{y}^{*|X}(x) = \overline{x}$ is then implemented when each agent reports his attribute truthfully. This implies

$$\widehat{r}_i^{a,*}(x_i) = x_i.$$



(a) $(\phi_a, [-6/5, 6/5])$ (b) $(\phi_m, R = [-3/4, 3/4])$

Figure 2: Difference in the severity of the exaggeration problem [$\alpha = 1/5$, $X_i \sim \mathrm{U}[-1, 1]$]

Each agent prefers however to exaggerate his attribute by the factor $N(1-\alpha) > 1$ in order to revert the effect of averaging. For any given alignment in preferences $\alpha \in [0, 1)$, the preferred degree of exaggeration increases unboundedly as N increases. This makes the conflict of interest between each agent and the social planner potentially very severe. See Figure 2a for an illustration. The gray curve depicts the socially optimal reporting behavior; the dotted curves indicate how the incentive to exaggerate becomes more severe as the number of agents increases; the black curves describe the actual reporting behavior for a specific report space.

The reason why an incentive to exaggerate arises also under the median aggregation rule is more subtle. It can be seen most easily for private preferences. Each agent prefers then to report his true attribute. When an agent with a high attribute learns that his attribute is the median attribute, he can infer that half of the other attributes must also be high and that the remaining attributes are either not just as high or much lower. Because this entails that he expects the higher attributes to lie closer to his own attribute than the lower attributes, he expects his own attribute to be higher than the average attribute. The bias of his individually preferred decision into the direction of his own attribute translates thus also under the median aggregation rule into an incentive to exaggerate.

To be more precise on in which sense exaggeration incentives arise, I need to derive the socially optimal reporting behavior. Under the median aggregation rule, the collective decision can depend only through the median attribute on the agents' information. The best decision conditional on this information $\widehat{y}^{*|\mathrm{m}(X)}(x) = \mathbf{E}_X[\overline{X}|\mathrm{m}(X) = \mathrm{m}(x)]$ is obtained when each agent reports his best estimate of the socially optimal decision $\overline{X}$ conditional on that his own attribute is the median attribute. This implies

$$\begin{aligned} \widehat{r}_i^{m,*}(x_i) &= \mathbf{E}_X[\overline{X}|\mathrm{m}(X_i, X_{-i}) = X_i, X_i = x_i] \\ &= 1/N \cdot x_i + (N-1)/N \cdot h(x_i). \end{aligned} \tag{3}$$

By comparing this reporting behavior with agent i's preferred reporting behavior $\widehat{r}_i^{m}(x_i)$, I obtain that exaggeration incentives arise for any imperfectly aligned preferences:

Lemma 3 *Consider any $N \in \mathcal{N}$ and any $\alpha \in [0,(N-1)/N)$. (a) $\widehat{r}_i^m(\underline{t}) < \widehat{r}_i^{m,*}(\underline{t})$ and $\widehat{r}_i^m(\overline{t}) > \widehat{r}_i^{m,*}(\overline{t})$. (b) If $F \in \mathcal{F}_h$, then there exists $t \in (\underline{t},\overline{t})$ such that $\widehat{r}_i^m(x_i) < \widehat{r}_i^{m,*}(x_i)$ for any $x_i < t$ and $\widehat{r}_i^m(x_i) > \widehat{r}_i^{m,*}(x_i)$ for any $x_i > t$. (c) If $F \in \mathcal{F}_h \cap \mathcal{F}_s$, $|\widehat{r}_i^{m,*}(x_i)| \leq |\widehat{r}_i^m(x_i)| \leq |x_i|$. (d) If $X_i \sim \mathrm{U}[-\overline{t},\overline{t}]$, $\widehat{r}_i^{m,*}(x_i) = (N+1)/(2N) \cdot x_i$ and $\widehat{r}_i^m(x_i) = (1-\alpha/2) \cdot x_i$.*

Both aggregation rules imply incentives to exaggerate relative to the socially optimal reporting behavior. The planner has thus for both aggregation rules to deal with an exaggeration problem. The nature of this problem differs however substantially. The most important difference is that under the median aggregation rule exaggeration incentives do not become more severe as the number of agents increases. The number of agents affects the socially optimal reporting behavior slightly because it affects the weight of any specific agent's attribute on the socially optimal decision $\overline{x}$. However, the dissent between the social planner and any agent is never very severe. See Figure 2b for an illustration and compare with Figure 2a.

5.2. The social planner's problem: How to deal with exaggeration?

Roughly speaking, the social planner faces the following problem: How to design the decision–making mechanism such that it allows the collective decision to be responsive to the agents' information without being too prone to strategic manipulations through exaggeration? Aggregation rule and report space represent two different instruments for restricting the agents' possibilities to manipulate the collective decision through strategic exaggerations. First, the designer can modify the aggregation rule by discarding more extreme reports. This restricts the impact of reports which are *extreme on a relative scale*. I study the trade–off implied by the two polar versions of this instrument where either no report is discarded or where all reports but the median report are discarded (see Subsection 7.1 for a discussion of intermediate trimming). Second, the designer can make the report space smaller by excluding extreme reports. This limits how *extreme* the submitted reports can be *on an absolute scale*.[18]

5.3. The reduced problem: Optimal delegation with a representative agent

I pose now a reduced–form version of the planner's problem to design the optimal report space for a given aggregation rule. Consider first $\phi = \phi_m$. I can use the socially optimal reporting behavior to rewrite welfare:[19]

$$V(\phi_m, R) = V^{*|\mathrm{m}(X)} + N \cdot \mathbf{E}_X[-(\widehat{r}_i(\mathrm{m}(X)) - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2]. \tag{4}$$

$V^{*|\mathrm{m}(X)}$ describes the unavoidable loss which arises because the median attribute $\mathrm{m}(x)$ is only imperfectly informative about the socially optimal decision $\overline{x}$. An additional loss which depends on the distance between the actual reporting behavior and the socially optimal reporting

[18]The designer may also have an incentive to restrict the report space for other reasons. For instance, she can make the induced decision locally more responsive to intermediate attribute realizations by forbidding intermediate reports. See Alonso and Matouschek (2008).

[19]See Appendix B for a derivation.

behavior arises when this imperfect information is not used optimally. Next, I can use (4) and the equilibrium reporting behavior in Proposition 2 (c) to formulate a reduced–form version of the planner's problem in which she faces a representative agent:

$$\begin{array}{ll} \max_R & V^{*|\mathrm{m}(X)} - N \cdot \mathbf{E}_{X_\mathrm{m}}[(\widehat{r}_i(X_\mathrm{m}) - \widehat{r}_i^{m,*}(X_\mathrm{m}))^2] \\ \text{s.t.} & \widehat{r}_i(x_\mathrm{m}) \in \arg\max_{r_i \in R} -(r_i - \widehat{r}_i^{m}(x_\mathrm{m}))^2. \end{array} \tag{5}$$

This problem can be interpreted as a standard delegation problem as considered in Holmström (1984) and in Alonso and Matouschek (2008). The role of the "delegation set" is thereby assumed by the report space; the role of the "decision" is assumed by the selected report. The representative agent is privately informed about the median attribute $X_\mathrm{m} := \mathrm{m}(X)$. His preferred decision is $\widehat{r}_i^{m}(x_\mathrm{m})$, whereas the social planner's preferred decision is $\widehat{r}_i^{m,*}(x_\mathrm{m})$.

Consider $\phi = \phi_a$. I can use again the socially optimal reporting behavior to rewrite welfare:[20]

$$V(\phi_a, R) = \mathbf{E}_{X_i}[-(\widehat{r}_i(X_i) - (\widehat{r}_i^{a,*}(X_i) - (\sqrt{N} - 1)b))^2] \tag{6}$$

with $b = \mathbf{E}_{X_i}[\widehat{r}_i(X_i)]$. A loss arises here from the distance between the equilibrium reporting behavior and the socially optimal reporting behavior corrected by a term which depends on the reporting bias. The dependence on this bias reflects that the social planner internalizes the detrimental effect of a biased reporting behavior on welfare. The socially optimal reporting behavior $\widehat{r}_i^{a,*}(x_i)$ is only the reporting behavior which the social planner prefers when reporting is indeed unbiased. Conditional on that the other agents' reporting behavior has a positive (resp. negative) bias, the social planner prefers a reporting behavior which is smaller (resp. larger) than $\widehat{r}_i^{a,*}(x_i)$ to correct for the bias. I can use (6) and the equilibrium reporting behavior in Proposition 1 (b) to formulate also for $\phi = \phi_a$ a reduced–form version of the planner's problem in which she faces a representative agent:

$$\begin{array}{ll} \max_R & \mathbf{E}_{X_i}[-(\widehat{r}_i(X_i) - (\widehat{r}_i^{a,*}(X_i) - (\sqrt{N} - 1)b))^2] \\ \text{s.t.} & \widehat{r}_i(x_i) \in \arg\max_{r_i \in R} -(r_i - \widehat{r}_i^{a}(x_i, b))^2 \\ & \mathbf{E}_{X_i}[\widehat{r}_i(X_i)] = b. \end{array} \tag{7}$$

The representative agent learns the realization of X_i as a private signal. His preferred "decision" is $\widehat{r}_i^{a}(x_i, b)$, whereas the social planner's preferred decision is $\widehat{r}_i^{a,*}(x_i) - (\sqrt{N} - 1)b$. Although the reduced–form problem resembles also in this case a delegation problem, the endogeneity of the parameter b which affects the planner's and the representative agent's preferred decision makes it non–standard.

The reduced problems allow me to interpret the role of the two instruments in the following way: The first instrument, the choice of the aggregation rule, simultaneously determines the quality of the representative agent's information and the severeness of the conflict of interest. Under the median aggregation rule there arises a welfare loss which cannot be avoided through the optimal use of the representative agent's information. This allows for the interpretation that

[20] See Appendix B for a derivation.

the quality of the representative agent's information is better under the average aggregation. On the other hand, the average aggregation rule comes often along with a very severe conflict of interest which cannot arise under the median aggregation rule. The second instrument, the design of the report space, can be used to mitigate the consequences of a given conflict of interest. Using it comes however at the cost that the decision reflects the representative agent's information less accurately.

Tools from the optimal delegation literature can be used to establish the existence of a solution to the reduced problems (see Theorem 1 in Holmström (1984)) and to derive the structure of the optimal solution (see Melumad and Shibano (1991) and Alonso and Matouschek (2008)). This literature is thus informative about the optimal use of my second instrument for a given use of the first instrument.[21] On the other hand, Grüner and Kiel (2004) study the optimal choice of the aggregation rule when reporting is unrestricted. Unless preferences are completely aligned, it is however for both aggregation rules optimal to impose a binding restriction on reporting.[22] Grüner and Kiel consider thus the optimal use of my first instrument for a given non–optimal use of my second instrument. Their analysis can be used as a benchmark case which helps me to assess the role of the second instrument.

6. Optimal information aggregation with strategic agents

At issue is now which aggregation rule the social planner prefers. I consider at first binary report spaces. This allows me to demonstrate that report spaces which allow only very coarse information to be reflected in the collective decision can often (but not always) be used as strong instruments to affect how the reflected information is used. I use then interval report spaces to motivate why it is less important for the planner to restrict reporting under the median aggregation rule. This endows me also with a basic understanding under which conditions the planner cannot do much better than choosing a binary report space and under which she can. Finally, I use the gained insights to analyze the choice between the aggregation rules for general report spaces.

6.1. Binary report spaces and the optimal aggregation of very coarse information

Suppose the report space is binary. Each agent is then basically asked to express whether he is leftist or rightist. The meaning of leftism and rightism is endogenous. It is determined by the

[21] Alonso and Matouschek (2008) discuss a committee decision problem with a given representative agent as an application of their theory on optimal delegation. The representative agent's private signal is distributed according to a truncated normal distribution. They discuss the optimal rules for the given representative agent (= the optimal report space), but they do not take into account that the nature of the representative agent is affected by the rules that the committee faces like the mode of aggregation.

[22] Responsible for this is that for both aggregation rules the agents' preferred reporting behavior exhibits exaggeration. By forbidding the most extreme reports, it is generally possible to bring the actual reporting behavior everywhere closer to the socially optimal reporting behavior. See Figure 2 for an illustration. The reporting behavior under the unrestricted mechanisms corresponds to the dotted curves. The gray areas depict specific report spaces which clearly enhance welfare.

threshold $t \in \mathcal{X}$ which separates who claims to be leftist and rightist. As it is not possible for an agent to express whether he is moderate or extreme, the exaggeration problem disappears. This comes at the cost that the collective decision can reflect only very coarse information about the agents' attributes. It can reflect the number of leftists and rightists if $\phi = \phi_a$ and whether the agent with the median attribute is leftist or rightist if $\phi = \phi_m$. The absence of the exaggeration problem gives the planner a degree of freedom regarding how the reflected information affects the collective decision. More technically, the average mechanism $(\phi_a, R^{a,B}(t,\delta))$ induces for any $\delta > 0$ the same meaning of leftism and rightism t. As the collective decision reflects for any δ the number of attributes which are below and above t, the planner can use δ to manipulate how the reflected information is used without affecting which information is reflected. An analogous reasoning applies for median mechanisms $(\phi_m, R^{m,B}(t,\delta))$.

Which aggregation rule does the planner prefer? When the minimal symmetry assumption $F \in \mathcal{F}_s$ and the regularity assumption $F \in \mathcal{F}_h$ is met, I get a clear and intuitive answer:

Proposition 3 *Let $F \in \mathcal{F}_h \cap \mathcal{F}_s$ and consider any $N \in \mathcal{N}$ and any $\alpha \in [0,(N-1)/N]$. Then, $\max_{R\in\mathcal{R}_B} V(\phi_a, R) > \max_{R\in\mathcal{R}_B} V(\phi_m, R)$. Moreover, $(\phi_a, \{\mathbf{E}_{X_i}[X_i|X_i \leq 0], \mathbf{E}_{X_i}[X_i|X_i \geq 0]\})$ is optimal among all binary average mechanisms.*

The reasoning relies on three properties: First, if $F \in \mathcal{F}_s$, any mechanism $(\phi, \{-\delta,\delta\})$ induces the same reporting threshold $t = 0$. If $\phi = \phi_a$, the collective decision reflects the number of attributes with a positive and a negative sign; if $\phi = \phi_m$, the collective decision reflects the sign of the median attribute. Second, for each of the two aggregation rules, the single instrument δ suffices to implement the decision which is optimal conditional on the reflected information: $(\phi_a, \{-\delta,\delta\})$ with $\delta = \mathbf{E}_{X_i}[X_i|X_i \geq 0]$ implements $\widehat{y}^{*|(\mathrm{sgn}(X_1),\ldots,\mathrm{sgn}(X_N))}$; $(\phi_m, \{-\delta,\delta\})$ with $\delta = \mathbf{E}_X[\mathrm{m}(X)|\mathrm{m}(X) \geq 0]$ implements $\widehat{y}^{*|\mathrm{sgn}(\mathrm{m}(X))}$. As $(\mathrm{sgn}(X_1),\ldots,\mathrm{sgn}(X_n))$ is a sufficient statistic for $\mathrm{sgn}(\mathrm{m}(X))$, it follows that the average aggregation rule outperforms the median aggregation rule for the considered class of report spaces. Third, $F \in \mathcal{F}_s$ and $F \in \mathcal{F}_h$ imply that some symmetric report space $\{-\delta,\delta\}$ is indeed optimal.

When $F \in \mathcal{F}_h \cap \mathcal{F}_s$ does not hold, the comparison of the aggregation rules is non–trivial for two reasons. First, given any reporting threshold t, the planner can choose only one of the two reports freely (see Corollaries 2 and 4). If $t \neq 0$ or $F \notin \mathcal{F}_s$, this imposes in general a binding constraint on how the collective decision can depend on the information which it reflects. Hence, although the information reflected by the mechanism $(\phi_a, R^{a,B}(t,\delta_a))$ is still a sufficient statistic for the information reflected by the mechanism $(\phi_m, R^{m,B}(t,\delta_m))$ for any t, any δ_a and any δ_m, it is a priori unclear how some non–optimal use of the former information compares with some non–optimal use of the latter. Second, it can be optimal to induce different reporting thresholds for the two aggregation rules. The task is thus not to compare different ways of aggregating the same information as in the symmetric case, but to compare different ways of aggregating different information.

Despite these technical difficulties, I can say how the aggregation rules compare when the number of agents is large. Although the planner is not able to use the information about the sign of each agent's attribute optimally when $F \notin \mathcal{F}_s$, she can approximate the optimal use of this information with binary average mechanisms when the committee size gets large:

Lemma 4 *For any* $\alpha \in [0, 1)$, $\lim_{N\to\infty} \max_{R\in\mathcal{R}_B} V(\phi_a, R) = V^{*|(\text{sgn}(X_1),\ldots,\text{sgn}(X_N))} = -\sigma^2 + c(0)$. *The mechanism* $(\phi_a, R^{a,B}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0]))$ *is asymptotically optimal within the class of binary average mechanisms.*

What drives this result? Interestingly, the severe form of exaggeration incentives which arise under the average aggregation rule when the number of agents is large facilitates the optimal use of the information about the signs of the agents' attributes. On the other hand, it avoids that any other information can be used in a reasonable way. An intuition follows from three observations: First, when the committee size is large, it is very detrimental for welfare when the average aggregation rule is used and the agents' reporting behavior is biased. The reporting bias implied by the optimal binary average mechanism must thus vanish as the committee size gets large. Second, when agent i expects the other agents' reporting behavior to be unbiased, the function $\widehat{r}_i^a(x_i, 0) = N(1-\alpha)x_i$ which describes his preferred reporting behavior has an intercept at zero and becomes infinitely steep as the committee size increases. Any report space $\{\underline{r}, \overline{r}\}$ with $\underline{r} < 0 < \overline{r}$ implies thus a reporting threshold close to zero. As this means that a reporting threshold close to zero comes for "free", the planner can use the two degrees of freedom which she has for determining the reporting threshold, the high report and the low report to fine–tune the reports. Third, the optimal use of the information about the sign of each agent's attribute implies indeed an unbiased reporting behavior.

Lemma 4 suggests that the optimal binary average mechanism performs relatively well for large committees even when the assumption $F \in \mathcal{F}_s$ which I use in Proposition 3 does not hold. The optimal binary median mechanism performs then, on the other hand, very poorly:

Lemma 5 *Let* $F \notin \mathcal{F}_s$ *and consider any* $\alpha \in [0, 1)$. *Then,* $\lim_{N\to\infty} \max_{R\in\mathcal{R}_B} V(\phi_m, R) = \lim_{N\to\infty} V^{*|0} = -\sigma^2$.

Intuitively, when the distribution is asymmetric with $\eta \neq 0$, it is for large N very likely that the planner wants something that is very different from what the median agent wants: the median attribute is then likely to lie close to η and I have $\lim_{N\to\infty} \widehat{r}_i^m(\eta) = (1-\alpha)\eta$ but $\lim_{N\to\infty} \widehat{r}_i^{m,*}(\eta) = 0$. Because of this severe dissent, the planner cannot do much better than always taking the best uninformed decision.

Combining Lemma 4 and Lemma 5 gives me the following comparison result:

Proposition 4 *Let* $F \notin \mathcal{F}_s$ *and consider any* $\alpha \in [0, 1)$. *Then,* $\lim_{N\to\infty} \max_{R\in\mathcal{R}_B} V(\phi_a, R) > \lim_{N\to\infty} \max_{R\in\mathcal{R}_B} V(\phi_m, R)$.

6.2. Interval report spaces and the optimal coarseness of aggregated information

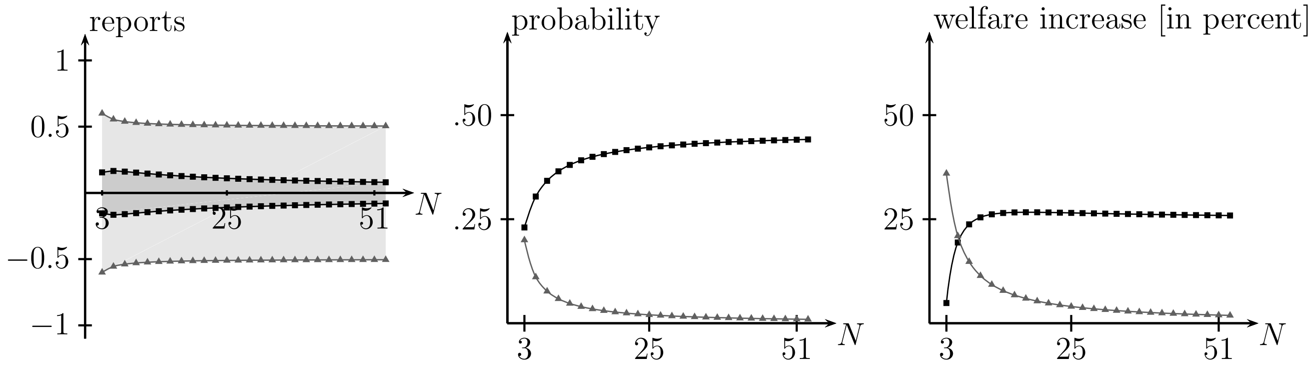
The role of the two elements of the decision–making mechanism is as follows: the aggregation rule determines which information *can be* aggregated/reflected in the collective decision; the report space determines simultaneously how coarsely the information that can be aggregated *is* aggregated/reflected in the collective decision and how this information is used. Which information can be aggregated is clearly better under the average aggregation rule. I explain now which implications the design of the report space has on the coarseness and the usage of the aggregated information. To obtain an intuition for the relevant effects, I consider at first interval report spaces. An agent is then to a certain extent able to express how moderate/extreme he is besides being able to express whether he is leftist or rightist.

Benchmark case: No binding restriction on reporting. Consider $R = \mathbb{R}$. The collective decision is then very responsive to the information which can be aggregated. At the same time, there are no constraints on how strongly an agent can exaggerate. This implies that the aggregated information is used poorly when the agents are eager to exaggerate strongly. To be more specific, consider what this means for each of the two aggregation rules. The collective decision implied by the average aggregation rule is $N(1-\alpha)\overline{x}$. It is perfectly correlated with the socially optimal decision $\overline{x}$. When the agents are eager to exaggerate strongly (which is the case when $N(1-\alpha)$ is large), the collective decision is nevertheless in most cases very bad. This is different when the median aggregation rule is used. The collective decision reflects then only the median agent's information, but the agents are never eager to exaggerate very strongly. It reflects thus worse information, but it uses this information never very poorly.

Lemma 6 *If $N(1-\alpha)$ is sufficiently large, then $V(\phi_a, \mathbb{R}) < V(\phi_m, \mathbb{R})$ and $V(\phi_a, \mathbb{R}) < V^{*|0}$.*

Grüner and Kiel (2004) establish in their main result that there is for any distribution and any number of agents a role for both aggregation rules when reporting is not restricted. The unrestricted average mechanism is superior when preferences are strongly aligned; the unrestricted median mechanism is superior for private preferences. Lemma 6 establishes that the superiority of the unrestricted average mechanism is for large N a peculiarity of almost perfectly aligned preferences: for any $\alpha \in [0, 1)$, the unrestricted median mechanism is superior when N is sufficiently large. Moreover, if the alignment in preferences is weak, the unrestricted average mechanism performs already very poorly for small committees. For example, it is for any $\alpha < 1/3$ worse than optimal centralized decision–making by an uninformed social planner even when there are only three committee members. This suggests that it is often important to restrict reporting significantly under the average aggregation rule.

A numerical example: Optimal interval report spaces. Figure 3 illustrates the optimal interval report spaces and properties thereof for private preferences, uniformly distributed attributes and different numbers of agents. Gray triangles indicate values for the average aggregation rule; black squares indicate values for the median aggregation rule. Figure 3a displays the optimal interval report spaces. The report space which is optimal under the average aggregation rule is a superset of $[-1/2, 1/2]$ which converges towards $[-1/2, 1/2]$ as the number of agents increases; the report space which is optimal under the median aggregation rule is for any number of agents much smaller and its length goes to zero as the number of agents increases.



(a) The optimal interval report space

(b) Optimal interval report space not binding

(c) Optimal interval versus optimal binary report space

Figure 3: Optimal interval report spaces for $\phi = \phi_a$ (gray triangles) and $\phi = \phi_m$ (black squares) [$X_i \sim \mathrm{U}[-1, 1]$, $\alpha = 0$]

What does this imply for the information that is reflected in the collective decision? Figure 3b illustrates for each reduced problem the probability with which the optimal interval report space does not impose a binding restriction on the representative agent. For interval report spaces, this probability serves as a reasonable measure for the accuracy with which the information that can be aggregated is aggregated. Although the optimal interval report space is always much larger under the average aggregation rule, the representative agent's information is reflected more accurately under the median aggregation rule. Because of the severe form of exaggeration incentives which arise under the average aggregation rule, only very coarse information is reflected in the collective decision even for moderate N. For large N, the collective decision reflects basically only binary information.[23] By contrast, the interval report space which is optimal for the median aggregation rule is non–binding with a non–vanishing probability. The information reflected in the collective decision is thus considerably more accurate than binary.

Figure 3c demonstrates by how much welfare increases when the optimal interval report space instead of the optimal binary report space is used. The relative welfare increase is for the average aggregation rule decreasing in the number of agents and it converges to 0%; for the median aggregation rule it is about 25% for most committee sizes. The planner's choice between the two aggregation rules involves thus a trade–off between using better information very coarsely ($\phi = \phi_a$) and using worse information more accurately ($\phi = \phi_m$).

The optimal coarseness of aggregated information. The numerical example provides an intuition for the coarseness of the aggregated information when attributes are uniformly distributed and the optimal interval report space is used. However, the optimal report space is in general not necessarily an interval (see the reasoning in Alonso and Matouschek (2008)) and there is often no good reason why the planner should not be able to choose a report space with a different

[23] A similar observation is made by De Sinopoli and Iannantuoni (2007) and Renault and Trannoy (2005) for an exogenously given finite report space and an exogenously given interval report space, respectively.

structure. The following lemma makes statements about the optimal coarseness of aggregated information for general distributions when the number of agents is large and I allow for general report spaces.

Lemma 7 *Consider* $\alpha \in [0, 1)$. *(a)* $\lim_{N\to\infty} \max_{R\in\mathcal{R}} V(\phi_a, R) = \lim_{N\to\infty} \max_{R\in\mathcal{R}_B} V(\phi_a, R)$. *(b) If* $F \notin \mathcal{F}_s$*, then* $\lim_{N\to\infty} \max_{R\in\mathcal{R}} V(\phi_m, R) = V^{*|0}$.

The numerical example demonstrated that the average mechanism with the optimal interval restriction may for large committees not perform significantly better than the optimal binary average mechanism. The driving force behind this, the severe form of exaggeration incentives under the average aggregation rule, depends neither on the employed distributional assumption nor on the restriction to interval report spaces. Part (a) of the lemma verifies that binary report spaces are indeed asymptotically optimal under the average aggregation rule. The numerical example demonstrated further that an analogous result cannot hold for the median aggregation rule.[24] Part (b) shows then that the optimal median mechanism can only do well in aggregating information for distributions which satisfy my minimal symmetry assumption. The same intuition as for the case with binary report spaces applies (see the discussion after Lemma 5).

6.3. The optimal aggregation rule for general report spaces

Which aggregation rule does the planner prefer when she can choose any report space? Answering this question is difficult for two reasons. First, using accurate information about the median attribute can for any number of agents be better than using coarse information about each agent's attribute. This is for example the case for uniformly distributed attributes:

Lemma 8 *Let* $X_i \sim \mathrm{U}[-\bar{t}, \bar{t}]$ *and consider any* $N \in \mathcal{N}$ *and any* $\alpha \in [0, (N-1)/N]$. *Then,* $V^{*|\mathrm{m}(X)} > V^{*|(\mathrm{sgn}(X_1),\ldots,\mathrm{sgn}(X_N))}$.

This makes it hard to derive bounds which prove the superiority of the average aggregation rule. Second, the structure of the optimal report space can be complicated and depends strongly on the specifics of the distribution. This makes it hard to derive results for general distributions. I need for these reasons to impose additional structure on the problem. I will first impose a distributional assumption and derive a result which holds for any number of agents. Afterwards, I will derive a result which holds for a general class of distributions when there is a large number of agents. Both results will hold for any degree of interdependence in preferences.

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[24] A formal proof of this statement can ensue the following reasoning: Consider attributes which are uniformly distributed on $[-1, 1]$. The binary report space which is optimal under the median aggregation rule is then symmetric about zero. It follows from Lemma 2 (e) and (d) that an upper bound on asymptotic welfare attained by any sequence of binary median mechanisms is $\lim_{N\to\infty} V^{*|\mathrm{sgn}(\mathrm{m}(X))} = \lim_{N\to\infty} V^{*|\mathrm{m}(X)} - (1 - 2/\pi)c(0)$. On the other hand, when I consider the specific sequence of non–binary median mechanisms $(\phi_m, [-2/(3\sqrt{N}), 2/(3\sqrt{N})])$ and use that the asymptotic distribution of $\sqrt{N}\mathrm{m}(X)$ is standard normal in (4), I obtain $\lim_{N\to\infty} V(\phi_m, [-2/(3\sqrt{N}), 2/(3\sqrt{N})]) = \lim_{N\to\infty} V^{*|\mathrm{m}(X)} - 2[\int_0^{2/3}((1-\alpha/2)z - 1/2 \cdot z)^2 \varphi(z)\mathrm{d}z + \int_{2/3}^{\infty}((1-\alpha/2)a - 1/2 \cdot z)^2]\varphi(z)\mathrm{d}z$ with $\varphi(z) = 1/\sqrt{2\pi} \cdot \exp(-z^2/2)$. Comparing these bounds proves then that there exist distributions $F \in \mathcal{F}_s$ such that $\lim_{N\to\infty} \max_{R\in\mathcal{R}} V(\phi_m, R) > \lim_{N\to\infty} \max_{R\in\mathcal{R}_B} V(\phi_m, R)$.

Comparison of the aggregation rules under specific distributional assumptions. Suppose the attributes are uniformly distributed.[25] I can then construct a specific average mechanism which performs better than the upper bound on welfare assumed by the optimal median mechanism, $V^{*|\mathrm{m}(X)}$. It follows from Lemma 8 that this requires the report space of the average mechanism to be non–binary. While this means that the optimal binary report space $\{-\bar{t}/2, \bar{t}/2\}$ does not do the job, I show in the proof of the following proposition that the interval report space $[-\bar{t}/2, \bar{t}/2]$ does.

Proposition 5 *Let $X_i \sim \mathrm{U}[-\bar{t}, \bar{t}]$ and consider any $N \in \mathcal{N}$ and any $\alpha \in [0, (N-1)/N]$. Then, $\max_{R\in\mathcal{R}} V(\phi_a, R) > \max_{R\in\mathcal{R}} V(\phi_m, R)$. The mechanism $(\phi_a, R^{a,I}(t', t''))$ with $t'' = \bar{t}/(2N(1-\alpha)-1)$ and $t' = -t''$ is optimal.*

Although the average aggregation rule implies a delegation problem with a representative agent which is non–standard (see (7)), the additional, non–standard constraint is not binding for symmetric distributions. The optimal report space follows from extending results from Melumad and Shibano (1991).

What is the role of the distributional assumption? As rough bounds suffice to prove the comparison result, the result is likely to extend to other distributions. My distributional assumption allows me to derive a tractable analytic expression for the upper bound on median welfare. When I derive this bound numerically, I can use my strategy of proof also to show the optimality of the average aggregation rule for other distributions. For instance, I can show that the result extends to the case with three agents, private preferences and distributions which have either a linear density or a density which is quadratic and symmetric around zero (see Appendix C for details).

Comparison of the aggregation rules for large numbers of agents. When the number of agents is large, I can use a different strategy of proof which allows me to dispose of the distributional assumption. I have then a lot of structure on the welfare assumed by the optimal average mechanism. This mechanism reflects basically only the very coarse information about the sign of each agent's attribute, but it uses this information optimally. Welfare converges to $V^{*|(\mathrm{sgn}(X_1),\ldots,\mathrm{sgn}(X_N))} = -\sigma^2 + c(0)$ as $N \to \infty$. By showing that the welfare assumed by the optimally restricted median mechanism converges to something which is strictly smaller than this, I obtain the following result:

Proposition 6 *Let either $F \in \mathcal{F}_s \cap \mathcal{F}_h$ or $F \notin \mathcal{F}_s$ and consider any $\alpha \in [0, 1)$. Then, $\lim_{N\to\infty} \max_{R\in\mathcal{R}} V(\phi_a, R) > \lim_{N\to\infty} \max_{R\in\mathcal{R}} V(\phi_m, R)$. Two asymptotically optimal mechanisms are $(\phi_a, R^{a,B}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0]))$ and $(\phi_a, R^{a,I}(t', t''))$ with $t'' = \mathbf{E}_{X_i}[X_i|X_i \geq 0]/(N(1-\alpha))$ and with t' being implicitly defined by $b^{a,I}(t', t'') = 0$.*

[25] The here considered uniform–quadratic case is widely discussed in the literature on delegation (e.g., Melumad and Shibano (1991)), collective decisions (e.g., Martimort and Semenov (2008)) and cheap talk (e.g., Crawford and Sobel (1982) and Kawamura (2011)).

The forces behind the result differ for $F \notin \mathcal{F}_s$ and $F \in \mathcal{F}_s \cap \mathcal{F}_h$. If $F \notin \mathcal{F}_s$, the result is not very surprising as for distributions with $\eta \neq 0$ and a large number of agents the optimal median mechanism performs very poorly (see Lemma 7 (b)). If $F \in \mathcal{F}_s \cap \mathcal{F}_h$, the result is less obvious as the optimally restricted median mechanism can then perform significantly better than the optimal binary median mechanism when the number of agents is large (see Figure 3c and Footnote 24). The comparison depends thus on how accurately the optimally restricted median mechanism reflects the median information. As the exact information about the median attribute is for large committees by Lemma 2 (d) as valuable as the information about the sign of each agent's attribute (that is, $\lim_{N\to\infty} V^{*|\mathrm{m}(X)} = -\sigma^2 + c(0)$), the proof reduces to showing that welfare assumed by the optimally restricted median mechanism is bounded away from $V^{*|\mathrm{m}(X)}$ for large N. Intuitively, this holds because the non–vanishing dissent between the agents and the social planner prevents an approximation of the optimal use of the median information.

Proposition 6 can be interpreted as good news for the practical design of decision–making processes which involve a large number of agents. Binary report spaces imply properties like strategic simplicity which may be desired in practice. Each agent's reporting problem boils down to deciding whether he is leftist or rightist. Moreover, the planner's problem to design the optimal report space becomes relatively simple as she does not have to deal with an exaggeration problem. The proposition establishes that also the performance of the optimal binary average mechanism is quite well when the number of agents is large.

7. Discussion of robustness

7.1. Intermediate trimming

In practice, trimmed mean mechanisms where an intermediate fraction $\rho \in (0, 1/2)$ of high and low reports is trimmed are often used. Consider for example the mechanism which is used for determining the LIBOR: the highest and the lowest quartile of reports is trimmed; the arithmetic mean of the remaining reports is selected as decision. This mechanism inherits properties from both classes of mechanisms studied in this article. Like for median mechanisms, an agent's (interim) preferred decision depends on what he can infer from his report not being trimmed. Like for average mechanisms, an agent exaggerates his (interim) preferred decision to counteract the effect of averaging. In particular, an agent's preferred degree of exaggeration increases for any $\rho \in (0, 1/2)$ unboundedly as the number of agents increases. Even though an exact equilibrium characterization is for intermediate trimming more involved than for the polar cases, the nature of the exaggeration incentives will also for intermediate trimming drive the asymptotic results. The severe form of exaggeration incentives will make a binary report space near optimal when the number of agents is large. For such report spaces, the exaggeration problem will basically disappear. However, given that the exaggeration problem disappears, there is no use in trimming as the only reason for trimming lies in mitigating exaggeration incentives. Hence, the average aggregation rule will also outperform any aggregation rule where an intermediate fraction of reports is trimmed when the number of agents is sufficiently large.

7.2. Non–quadratic utility

For any given welfare functional $V(\cdot)$, quadratic utility improves the tractability of the planner's design problem strongly, but important properties of equilibrium reporting behavior are robust with respect to the utility specification. To illustrate this, suppose preferences are private and each agent minimizes the absolute instead of the quadratic loss. That is, $u(y - \theta(x_i, x_{-i})) := -|y - x_i|$.[26] Each agent strives then for equalizing the probability of a positive and a negative loss. When the median aggregation rule is used, agent i's preferred report is x_i and he submits the admissible report which is closest to this. When the average aggregation rule is used, agent i strives for equalizing $\text{Prob}_X[1/N \cdot (r_i + \sum_{j \neq i} \widehat{r}_j(X_j)) \leq x_i]$ and $\text{Prob}_X[1/N \cdot (r_i + \sum_{j \neq i} \widehat{r}_j(X_j)) \geq x_i]$. If the distribution of $\widehat{r}_j(X_j)$ is symmetric about zero, his preferred report is Nx_i and he submits the closest admissible report. As restricting attention to the case in which $\widehat{r}_j(X_j)$ is symmetrically distributed is without loss of generality when the distribution of attributes is symmetric about zero, I obtain basically the same reduced problem as in the case with a quadratic loss function. Hence, all my results extend for private preferences and symmetrically distributed attributes to the case with an absolute loss function.[27]

7.3. The planner strives for implementing the median attribute

When the utility function $u(\cdot)$ is quadratic, a utilitarian planner minimizes the quadratic distance between the decision and the average attribute. Suppose now the planner strives for minimizing the quadratic distance to the median attribute instead. That is, consider $V(\widehat{y}) := N \cdot \mathbf{E}_X[u(\widehat{y}(X) - \text{m}(X))]$. Unlike for the case with a utilitarian social planner, the unrestricted median mechanism implements then the first–best when preferences are private ($\alpha = 0$). However, like for the case with a utilitarian social planner, the optimally restricted median mechanism performs very poorly for any intermediate degree of interdependence $\alpha \in (0, 1)$ when $\eta \neq 0$ and the number of agents is sufficiently large. It is then very likely that the socially optimal decision lies close to η, whereas the for the median agent individually optimal decision lies close to $(1-\alpha)\eta$. Hence, by a similar reasoning as in Lemma 5, the optimally restricted median mechanism cannot perform significantly better than optimal centralized decision–making by an uninformed social planner. On the hand, by a reasoning similar to that which I give

[26] See Section 5 in Grüner and Kiel (2004) for a related discussion for the case in which reporting is unrestricted.

[27] When the distribution of attributes is non–symmetric, the problem becomes technically more involved. Agent i's preferred report under the average aggregation rule is then harder to specify and he does not necessarily choose the closest admissible report when his preferred report is not admissible. The general trade–off does however survive: the average aggregation rule can reflect better information but it comes along with a conflict of interest which becomes unboundedly severe as the number of agents increases; the median aggregation rule can reflect only worse information, but the conflict of interest is never very severe. A similar reasoning can be applied to argue that the same general trade–off arises also for non–private preferences and other loss functions.

along Lemma 4, binary average mechanisms can perform quite well for large committees even when the distribution is asymmetric.[28]

8. Conclusion

For an environment in which strategically acting agents with interdependent preferences and diverse private information have to take a continuous collective decision, I studied decision–making mechanisms from the viewpoint of a utilitarian social planner. The planner decided on whether the collective decision is the average or the median report, and on the set of reports which the agents could submit. For the much discussed case with uniformly distributed information, I found that the average aggregation rule outperforms the median aggregation rule for any degree of interdependence and for any number of agents when the report space is optimally designed. The result extends to general distributions of information when the number of agents is large. Moreover, eliciting binary information from each agent is then near optimal under the average aggregation rule.

My analysis shedded light on the trade–offs associated with the design of trimmed mean mechanisms. It adds thus to the better understanding of a class off collective decision–making mechanisms which is for its simplicity and transparency important in practice.[29] A comparison of my results with those in Grüner and Kiel (2004) reveals that the desirability of trimming depends strongly on the planner's ability to restrict reporting.

My analysis established further that there exists a link between collective decision–making and optimal delegation. I showed that the design of the report space for a given aggregation rule corresponds to a delegation problem with a representative agent. The representative agent's preferences derive endogenously and depend on the aggregation rule. The choice between different aggregation rules, which was the focus of my study, can thus be interpreted as the choice between delegation problems with different representative agents. My analysis has two important implications: On the one hand, it shows how the optimal delegation literature can help to better understand collective decision problems. In that respect, it complements the seminal work by Alonso and Matouschek (2008) about optimal delegation with an exogenously given agent. On the other hand, it demonstrates that applying results from the optimal dele-

[28] There are two main differences to the result in Lemma 4. First, the optimal binary average mechanism does not necessarily extract the information about the sign of each agent's attribute, but about whether each agent's attribute exceeds some other threshold. Second, the bound towards which welfare of the optimal binary average mechanism converges is harder to quantify.

[29] Average and median mechanisms serve also as a modeling tool for different institutions. Example 1: pluralistic versus proportional representation. The outcome of majority voting in a one–dimensional environment with single–peaked preferences is normally the median voter's preferred decision (see the literature following Black (1948)), whereas the average aggregation rule resembles proportional representation (Ortuño–Ortín (1997), De Sinopoli and Iannantuoni (2007)). Example 2: joint responsibility versus individual responsibility. Consider an investment decision made by managers in a firm. Each manager may independently decide on one part of the investment up to a certain amount (see Section 3 in Holmström (1984) for such an interpretation) or the managers can jointly decide with the median proposal being selected as investment.

gation literature naively can lead to misleading conclusions: Although it may sometimes seem tempting to assume that a group of agents is represented by its median agent and to apply the results of Alonso and Matouschek directly, the median agent does normally not represent how a group decides when the mode of information aggregation is optimally designed.[30]

One of the main lessons of this article is that the average aggregation rule (without trimming) performs well among simple aggregation rules. Finding the generally optimal mode of aggregation would be interesting, but Bayesian implementation in the absence of monetary incentives makes this a hard problem already for two agents and private preferences (see Carrasco and Fuchs (2009)). I leave further investigations in that direction for future research.

Appendix A. Auxiliary results

Lemma A1 *(a)* $h(\underline{t}) = \underline{t}/2$, $h(\eta) = 0$ *and* $h(\overline{t}) = \overline{t}/2$. *(b) If* $F \in \mathcal{F}_h$, *then* $h'(x_i) \in (0,1)$. *(c)* $h'(x_i)$ *is continuous at* $x_i = \eta$ *with* $h'(\eta) = 2f(\eta)\mathbf{E}_{X_i}[X_i|X_i \geq \eta]$. *(d) If* $X_i \sim \mathrm{U}[-\overline{t},\overline{t}]$, *then* $h(x_i) = x_i/2$.

Proof. (a) The first (resp. last) property follows from $h(\underline{t}) = (\underline{t} + \mathbf{E}_{X_j}[X_j])/2$ (resp. $h(\overline{t}) = (\mathbf{E}_{X_j}[X_j]+\overline{t})/2$) and the normalization of the distribution. To obtain the second property, I can use $F(\eta) = 1/2$ to write $h(\eta) = F(\eta)\mathbf{E}_{X_j}[X_j|X_j \leq \eta] + (1 - F(\eta))\mathbf{E}_{X_j}[X_j|X_j \geq \eta] = \mathbf{E}_{X_j}[X_j]$. The normalization of the distribution implies then also the second property.

(b) I have

$$h(x_i) = \frac{1}{2}\left(\frac{\int_{\underline{t}}^{x_i} x_j f(x_j)\mathrm{d}x_j}{F(x_i)} + \frac{\int_{x_i}^{\overline{t}} x_j f(x_j)\mathrm{d}x_j}{1-F(x_i)}\right) \tag{A.1}$$

$$= x_i + \frac{1}{2}\left(-\frac{\int_{\underline{t}}^{x_i} F(x_j)\mathrm{d}x_j}{F(x_i)} + \frac{\int_{x_i}^{\overline{t}}(1-F(x_i))\mathrm{d}x_j}{1-F(x_i)}\right). \tag{A.2}$$

The first equality follows from rewriting the expected value expressions in the definition of $h(x_i)$ as integrals. The second equality follows from applying integration by parts.

By differentiating (A.2), I get

$$\begin{aligned}
h'(x_i) &= 1 + \frac{1}{2}\left(-\frac{F(x_i)^2 - \int_{\underline{t}}^{x_i} F(x_j)\mathrm{d}x_j f(x_i)}{F(x_i)^2}\right. \\
&\qquad \left. + \frac{-(1-F(x_i))^2 + \int_{x_i}^{\overline{t}}(1-F(x_j))\mathrm{d}x_j f(x_i)}{(1-F(x_i))^2}\right) \\
&= \frac{1}{2}\left(\frac{1}{F(x_i)}\int_{\underline{t}}^{x_i}\frac{f(x_i)}{F(x_i)}F(x_j)\mathrm{d}x_j + \frac{1}{1-F(x_i)}\int_{x_i}^{\overline{t}}\frac{f(x_i)}{1-F(x_i)}(1-F(x_j))\mathrm{d}x_j\right) \\
&= \frac{1}{2}\left(\mathbf{E}_{X_j}\left[\left.\frac{f(x_i)}{F(x_i)}\middle/\frac{f(X_j)}{F(X_j)}\right| X_j \leq x_i\right] + \mathbf{E}_{X_j}\left[\left.\frac{f(x_i)}{1-F(x_i)}\middle/\frac{f(X_j)}{1-F(X_j)}\right| X_j \geq x_i\right]\right).
\end{aligned}$$

[30] For example, see the reference in Footnote 21.

Since this is clearly positive, $h'(x_i) > 0$. Since f/F is strictly decreasing and $f/(1-F)$ is strictly increasing for $F \in \mathcal{F}_h$, the two expected value expressions are weakly smaller than one. Since at least one of them must be strictly smaller, I obtain $h'(x_i) < 1$.

(c) By differentiating (A.1), I get

$$\begin{aligned} h'(x_i) &= \frac{1}{2}\left(\frac{x_i f(x_i)F(x_i) - f(x_i)\int_{\underline{t}}^{x_i} x_j f(x_j)\mathrm{d}x_j}{F(x_i)^2} \right. \\ &\qquad \left. + \frac{-x_i f(x_i)(1-F(x_i)) + f(x_i)\int_{x_i}^{\overline{t}} x_j f(x_j)\mathrm{d}x_j}{(1-F(x_i))^2} \right) \\ &= \frac{1}{2}\left(x_i \left(\frac{f(x_i)}{F(x_i)} - \frac{f(x_i)}{1-F(x_i)} \right) \right. \\ &\qquad \left. - \frac{f(x_i)}{F(x_i)}\mathbf{E}_{X_j}[X_j|X_j \leq x_i] + \frac{f(x_i)}{1-F(x_i)}\mathbf{E}_{X_j}[X_j|X_j \geq x_i] \right). \end{aligned} \tag{A.3}$$

The second equality follows from simplifying and from writing the integrals as expected values.

Since $h'(x_i)$ is a composition of continuous functions, it is continuous. By evaluating (A.3) at η, I obtain $h'(\eta) = f(\eta)(-\mathbf{E}_{X_j}[X_j|X_j \leq \eta] + \mathbf{E}_{X_j}[X_j|X_j \geq \eta])$. Since a consequence of $h(\eta) = 0$ (see Part (a)) is that $\mathbf{E}_{X_i}[X_i|X_i \leq \eta] = -\mathbf{E}_{X_i}[X_i|X_i \geq \eta]$, I obtain $h'(\eta) = 2f(\eta)\mathbf{E}_{X_i}[X_i|X_i \geq \eta]$.

(d) Since $\mathbf{E}_{X_j}[X_j|X_j \geq x_i] = (x_i + \overline{t})/2$ and $\mathbf{E}_{X_j}[X_j|X_j \leq x_i] = (-\overline{t} + x_i)/2$ for uniformly distributed attributes, I get $h(x_i) = x_i/2$. q.e.d.

Appendix B. Proofs

Proof of the decomposition of the planner's expected utility in Section 3.

I have

$$\begin{aligned} U_0(\widehat{y}) &= \mathbf{E}_X[\textstyle\sum_i u(\widehat{y}(X) - \theta(X_i, X_{-i}))] \\ &= \textstyle\sum_i \mathbf{E}_X[-(\widehat{y}(X) - \overline{X})^2 - 2(\widehat{y}(X) - \overline{X})(\overline{X} - \theta(X_i, X_{-i})) - (\overline{X} - \theta(X_i, X_{-i}))^2] \\ &= N \cdot \mathbf{E}_X[-(\widehat{y}(X) - \overline{X})^2] + 0 + N \cdot \mathbf{E}_X[-(\overline{X} - \theta(X_i, X_{-i}))^2] \\ &= V(\widehat{y}) + \overline{U}_0. \end{aligned}$$

The second equality follows from adding and subtracting $\overline{X}$, and from multiplying out. The third equality follows from using $\sum_i(\overline{X} - \theta(X_i, X_{-i})) = 0$ and that attributes are identically distributed. q.e.d.

Proof of Lemma 1.

(a) For any $x \in \mathcal{X}^N$, the social planner chooses $y \in \mathbb{R}$ to maximize

$$\begin{aligned} & N \cdot \mathbf{E}_X[u(y - \overline{X})|\mathrm{Stat}(X) = \mathrm{Stat}(x)] \\ = \;& N \cdot \mathbf{E}_X[-(y - \mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)])^2|\mathrm{Stat}(X) = \mathrm{Stat}(x)] \\ & +N \cdot \mathbf{E}_X[-2(y - \mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)])(\mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)] - \overline{X})|\mathrm{Stat}(X) = \mathrm{Stat}(x)] \\ & +N \cdot \mathbf{E}_X[-(\mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)] - \overline{X})^2|\mathrm{Stat}(X) = \mathrm{Stat}(x)] \\ = \;& -N \cdot (y - \mathbf{E}_X[\overline{X}|\mathrm{Stat}(X) = \mathrm{Stat}(x)])^2 \\ & -N \cdot \mathbf{E}_X[(\mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)] - \overline{X})^2|\mathrm{Stat}(X) = \mathrm{Stat}(x)] \end{aligned}$$

The transformations arise as follows: The first equality follows from using $y - \overline{X} = (y - \mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)]) + (\mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)] - \overline{X})$, and from multiplying out. The second equality follows from pulling expressions which depend only through $\mathrm{Stat}(X)$ on X out of the expected value expressions, and from using that the second expected value expression is zero by the Law of Iterated Expectations.

Since the social planner's objective function depends only through the term $-N \cdot (y - \mathbf{E}_X[\overline{X}|\mathrm{Stat}(X) = \mathrm{Stat}(x)])^2$ on the collective decision, $y = \mathbf{E}_X[\overline{X}|\mathrm{Stat}(X) = \mathrm{Stat}(x)]$ is optimal for her. This proves that $\widehat{y}^{*|\mathrm{Stat}(X)}(x)$ is as claimed in the text before the lemma. I obtain

$$\begin{aligned} V(\widehat{y}^{*|\mathrm{Stat}(X)}) &= -N \cdot \mathbf{E}_X[(\mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)] - \overline{X})^2] \\ &= -N \cdot \mathbf{E}_X[\mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)]^2 - 2\mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)]\overline{X} + \overline{X}^2] \\ &= -\sigma^2 + N \cdot \mathbf{E}_X[\mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)]^2]. \end{aligned}$$

The third equality follows from using two things. First, by the Law of Iterated Expectations, $\mathbf{E}_X[\mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)]\overline{X}] = \mathbf{E}_X[\mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)]^2]$. Second, since attributes are independently and identically distributed, $N\mathbf{E}_X[\overline{X}^2] = \sigma^2$.

(b) If $\mathrm{Stat}(X) = (\mathrm{S}(X_1), \ldots, \mathrm{S}(X_N))$, I can simplify the formula in Part (a) as follows:

$$\begin{aligned} & V(\widehat{y}^{*|\mathrm{Stat}(X)}) \\ = \; & -\sigma^2 + 1/N \cdot \mathbf{E}_X[(\textstyle\sum_i \mathbf{E}_X[X_i|\mathrm{S}(X_i)])^2] \\ = \; & -\sigma^2 + 1/N \cdot \textstyle\sum_i \Big(\mathbf{E}_X[\mathbf{E}_X[X_i|\mathrm{S}(X_i)]^2] + \sum_{j\neq i} \mathbf{E}_X[\mathbf{E}_X[X_i|\mathrm{S}(X_i)]\mathbf{E}_X[X_j|\mathrm{S}(X_j)]]\Big) \\ = \; & -\sigma^2 + \mathbf{E}_X[\mathbf{E}_X[X_i|\mathrm{S}(X_i)]^2] + \textstyle\sum_{j\neq i} \mathbf{E}_X[\mathbf{E}_X[X_i|\mathrm{S}(X_i)]] \cdot \mathbf{E}_X[\mathbf{E}_X[X_j|\mathrm{S}(X_j)]] \\ = \; & -\sigma^2 + \mathbf{E}_{X_i}[\mathbf{E}_{X_i}[X_i|\mathrm{S}(X_i)]^2] \end{aligned}$$

The first equality follows from using that $\mathbf{E}_X[\overline{X}|\mathrm{Stat}(X)] = 1/N \cdot \sum_i \mathbf{E}_X[X_i|\mathrm{S}(X_i)]$. The second equality follows from multiplying out. The third equality follows from using that attributes are independently and identically distributed. The fourth equality follows from using that $\mathbf{E}_X[\mathbf{E}_X[X_i|\mathrm{S}(X_i)]] = \mathbf{E}_X[X_i]$ by the Law of Iterated Expectations and that $\mathbf{E}_X[X_i] = 0$ by the normalization of the distribution. q.e.d.

Proof of Lemma 2.

The statistic of X which the social planner learns in Parts (a), (b) and (c) has the structure $\mathrm{Stat}(X) = (\mathrm{S}(X_1), \ldots, \mathrm{S}(X_n))$ for some function S such that Lemma 1 (b) applies. (a) I have $\mathrm{S}(X_i) = X_i$ such that $\mathbf{E}_{X_i}[X_i|\mathrm{S}(X_i)] = X_i$. This implies $V^{*|X} = -\sigma^2 + \mathbf{E}_{X_i}[X_i^2] = 0$. (b) I have $\mathrm{S}(X_i) = \mathrm{sgn}(X_i - t)$. Since this implies $\mathbf{E}_{X_i}[\mathbf{E}_{X_i}[X_i|\mathrm{S}(X_i)]^2] = c(t)$, I obtain $V^{*|(\mathrm{sgn}(X_1-t),\ldots,\mathrm{sgn}(X_N-t))} = -\sigma^2 + c(t)$. (c) I have $\mathrm{S}(X_i) = 0$ such that $\mathbf{E}_{X_i}[X_i|\mathrm{S}(X_i)] = 0$ by the normalization of the distribution. This implies $V^{*|0} = -\sigma^2 + 0$.

(d) I have

$$\begin{aligned} V^{*|\mathrm{m}(X)} &= -\sigma^2 + N \cdot \mathbf{E}_X[\mathbf{E}_X[\overline{X}|\mathrm{m}(X)]^2] \\ &= -\sigma^2 + 1/N \cdot \mathbf{E}_X[(\mathrm{m}(X) + (N-1)h(\mathrm{m}(X)))^2] && \text{(B.1)} \\ &= -\sigma^2 + 1/N^2 \cdot \mathbf{E}_X[(\sqrt{N}\mathrm{m}(X))^2] + (N-1)^2/N^2 \cdot \mathbf{E}_X[(\sqrt{N}h(\mathrm{m}(X)))^2] \\ &\qquad +2(N-1)/N \cdot \mathbf{E}_X[\mathrm{m}(X)h(\mathrm{m}(X))]. && \text{(B.2)} \end{aligned}$$

The first equality follows from Lemma 1 (a). The second equality follows from using that $\mathbf{E}_X[\overline{X}|\mathrm{m}(X)] = 1/N \cdot (\mathrm{m}(X) + (N-1)h(\mathrm{m}(X)))$. The third equality follows from multiplying out.

By the Corollary to Theorem 13 in Ferguson (1996), the asymptotic distribution of $\sqrt{N} \cdot (\mathrm{m}(X) - \eta)$ is $\mathrm{N}(0, 1/(2f(\eta))^2)$. For $F \in \mathcal{F}_s$ this has the following three implications: First, $\lim_{N\to\infty} \mathbf{E}_X[(\sqrt{N}\mathrm{m}(X))^2] = 1/(2f(0))^2$ such that $\lim_{N\to\infty} 1/N^2 \cdot \mathbf{E}_X[(\sqrt{N}\mathrm{m}(X))^2] = 0$. Second, by using that $h(0) = 0$ (see Lemma A1 (a) in Appendix A) and that $h'(x_i)$ is continuous at 0 with $h'(0) = 2f(0)\mathbf{E}_{X_i}[X_i|X_i \geq 0]$ (see Lemma A1 (c) in Appendix A), I can apply Theorem 7 in Ferguson (1996) to obtain that the asymptotic distribution of $\sqrt{N}h(\mathrm{m}(X))$ is $\mathrm{N}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0]^2)$. This implies $\lim_{N\to\infty}(N-1)^2/N^2 \cdot \mathbf{E}_X[(\sqrt{N}h(\mathrm{m}(X)))^2] = \mathbf{E}_{X_i}[X_i|X_i \geq 0]^2$. Third, $\lim_{N\to\infty} 2(N-1)/N \cdot \mathbf{E}_X[\mathrm{m}(X)h(\mathrm{m}(X))] = 0$.

From (B.2) and the three implications, I obtain

$$\lim_{N\to\infty} V^{*|\mathrm{m}(X)} = -\sigma^2 + \mathbf{E}_{X_i}[X_i|X_i \geq 0]^2.$$

Since $h(0) = 0$ holds for $F \in \mathcal{F}_s$ (see Lemma A1 (a) in Appendix A) and since this implies $\mathbf{E}_{X_i}[X_i|X_i \leq 0] = -\mathbf{E}_{X_i}[X_i|X_i \geq 0]$, I can write $\mathbf{E}_{X_i}[X_i|X_i \geq 0]^2 = F(0)\mathbf{E}_{X_i}[X_i|X_i \leq 0]^2 + (1 - F(0))\mathbf{E}_{X_i}[X_i|X_i \geq 0]^2$. Since this corresponds to $c(0)$, I obtain the result.

(e) Let $\mathrm{S}(x_i) = 1$ if $x_i \geq 0$ and $\mathrm{S}(x_i) = 0$ if $x_i < 0$. I have then

$$\begin{aligned}
& V^{*|\mathrm{sgn}(\mathrm{m}(X))} \\
= \;& V^{*|\mathrm{sgn}(\sum_i \mathrm{S}(X_i) - N/2)} \\
= \;& -\sigma^2 + N \cdot \mathbf{E}_X[\mathbf{E}_X[\overline{X}|\mathrm{sgn}(\textstyle\sum_i \mathrm{S}(X_i) - N/2)]^2] \\
= \;& -\sigma^2 + N \cdot \mathbf{E}_X[\mathbf{E}_X[\mathbf{E}_X[\overline{X}|\textstyle\sum_i \mathrm{S}(X_i) - N/2]|\mathrm{sgn}(\sum_i \mathrm{S}(X_i) - N/2)]^2] \\
= \;& -\sigma^2 + N \cdot \mathbf{E}_X[\mathbf{E}_X[1/N \cdot ((\textstyle\sum_i \mathrm{S}(X_i)) \cdot \mathbf{E}_{X_i}[X_i|X_i \geq 0] \\
& \quad + (N - \textstyle\sum_i \mathrm{S}(X_i)) \cdot \mathbf{E}_{X_i}[X_i|X_i < 0])|\mathrm{sgn}(\sum_i \mathrm{S}(X_i) - N/2)]^2] \\
= \;& -\sigma^2 + N \cdot \mathbf{E}_X[\mathbf{E}_X[2(\textstyle\sum_i \mathrm{S}(X_i) - N/2)/N \cdot \mathbf{E}_{X_i}[X_i|X_i \geq 0]|\mathrm{sgn}(\sum_i \mathrm{S}(X_i) - N/2)]^2] \\
= \;& -\sigma^2 + \mathbf{E}_X[\mathbf{E}_X[(\textstyle\sum_i \mathrm{S}(X_i) - N/2)/\sqrt{N/4}|\mathrm{sgn}(\sum_i \mathrm{S}(X_i) - N/2)]^2] \cdot \mathbf{E}_{X_i}[X_i|X_i \geq 0]^2 \\
= \;& -\sigma^2 + \mathbf{E}_X[(\textstyle\sum_i \mathrm{S}(X_i) - N/2)/\sqrt{N/4}|\sum_i \mathrm{S}(X_i) - N/2 \geq 0]^2 \cdot \mathbf{E}_{X_i}[X_i|X_i \geq 0]^2 \\
= \;& -\sigma^2 + \mathbf{E}_X[|(\textstyle\sum_i \mathrm{S}(X_i) - N/2)/\sqrt{N/4}|]^2 \cdot \mathbf{E}_{X_i}[X_i|X_i \geq 0]^2.
\end{aligned}$$

The transformations arise as follows: The first equality follows from using that $\mathrm{sgn}(\mathrm{m}(X)) = \mathrm{sgn}(\sum_i \mathrm{S}(X_i) - N/2)$. The second equality follows from applying Lemma 1 (a). The third equality follows from applying the Law of Iterated Expectations. The fourth equality follows from using that $\mathbf{E}_X[\overline{X}|\sum_i \mathrm{S}(X_i) - N/2] = 1/N \cdot ((\sum_i \mathrm{S}(X_i)) \cdot \mathbf{E}_{X_i}[X_i|X_i \geq 0] + (N - \sum_i \mathrm{S}(X_i)) \cdot \mathbf{E}_{X_i}[X_i|X_i < 0]$ for independently and identically distributed attributes. The fifth equality follows from using that $F \in \mathcal{F}_s$ implies $h(0) = 0$ (see Lemma A1 (a) in Appendix A) which in turn implies $\mathbf{E}_{X_i}[X_i|X_i < 0] = -\mathbf{E}_{X_i}[X_i|X_i \geq 0]$. The sixth equality follows from simplifying. Note that $\sum_i \mathrm{S}(X_i)$ is binomially distributed with success probability 1/2 and number of trials N. This implies that the distribution of $\sum_i \mathrm{S}(X_i) - N/2$ is symmetric about zero such that $\mathbf{E}_X[\sum_i \mathrm{S}(X_i) - N/2|\sum_i \mathrm{S}(X_i) - N/2 \geq 0]^2 = \mathbf{E}_X[\sum_i \mathrm{S}(X_i) - N/2|\sum_i \mathrm{S}(X_i) - N/2 \leq 0]^2$. By using this, I obtain the seventh equality. Finally, the eight equality follows from using again the symmetry of the distribution of $\sum_i \mathrm{S}(X_i) - N/2$.

Because $\sum_i \mathrm{S}(X_i)$ is binomially distributed with mean $N/2$ and variance $N/4$, the asymptotic distribution of $(\sum_i \mathrm{S}(X_i) - N/2)/(\sqrt{N/4})$ is normal with mean zero and variance 1. This implies that the asymptotic distribution of $|(\sum_i \mathrm{S}(X_i) - N/2)/(\sqrt{N/4})|$ is half–normal with mean $\sqrt{2/\pi}$. Hence,

$$\lim_{N\to\infty} V^{*|\mathrm{sgn}(\mathrm{m}(X))} \;=\; -\sigma^2 + 2/\pi \cdot \mathbf{E}_{X_i}[X_i|X_i \geq 0]^2.$$

Since $\mathbf{E}_{X_i}[X_i|X_i \geq 0]^2 = c(0)$ by the reasoning at the end of Part (d), I obtain the result.

(f) Consider $F(x_i) = (x_i + \bar{t})/(2\bar{t})$. I have then

$$\begin{aligned} V^{*|\mathrm{m}(X)} &= -\sigma^2 + 1/N \cdot \mathbf{E}_X[(\mathrm{m}(X) + (N-1)h(\mathrm{m}(X)))^2] \\ &= -\sigma^2 + (N+1)^2/(4N) \cdot \mathbf{E}_X[\mathrm{m}(X)^2] \end{aligned} \tag{B.3}$$

The first equality follows from (B.1) since $F \in \mathcal{F}_s$. The second equality follows from using that $h(x_i) = x_i/2$ (see Lemma A1 (d) in Appendix A) and from simplifying.

It remains to derive an algebraic expression for $\mathbf{E}_X[\mathrm{m}(X)^2]$. Since the median order statistic of N independently and identically distributed signals is distributed according to the density function $f_\mathrm{m}(x_\mathrm{m}) := N!/(M-1)!^2 \cdot F(x_\mathrm{m})^{M-1} f(x_\mathrm{m})(1-F(x_\mathrm{m}))^{M-1}$ with $M := (N+1)/2$, I can write

$$\begin{aligned} & \mathbf{E}_X[\mathrm{m}(X)^2] \\ = \; & \textstyle\int_{-\bar{t}}^{\bar{t}} x_\mathrm{m}^2 f_\mathrm{m}(x_\mathrm{m}) \mathrm{d}x_\mathrm{m} \\ = \; & \textstyle\int_0^1 F^{-1}(v)^2 f_\mathrm{m}(F^{-1}(v)) \cdot (1/f(F^{-1}(v))) \cdot \mathrm{d}v \\ = \; & N!/(M-1)!^2 \cdot \textstyle\int_0^1 (2v-1)^2 v^{M-1}(1-v)^{M-1} \mathrm{d}v \cdot \bar{t}^2 \\ = \; & N!/(M-1)!^2 \cdot (4\mathrm{Beta}(M+2,M) - 4\mathrm{Beta}(M+1,M) + \mathrm{Beta}(M,M)) \cdot \bar{t}^2 \\ = \; & \frac{N!}{(M-1)!^2} \cdot \left(4\frac{(M+1)!(M-1)!}{(2M+1)!} - 4\frac{(M)!(M-1)!}{(2M)!} + \frac{(M-1)!(M-1)!}{(2M-1)!}\right) \cdot \bar{t}^2 \\ = \; & \left(4\frac{(M+1)M}{(2M+1)2M} - 4\frac{M}{2M} + \frac{1}{1}\right) \cdot \bar{t}^2 \\ = \; & 1/(N+2) \cdot \bar{t}^2. \end{aligned} \tag{B.4}$$

The transformations arise as follows: The second equality follows from substituting $x_\mathrm{m} = F^{-1}(v)$. The third equality follows from using that $F^{-1}(v) = (2v-1)\bar{t}$, from using the definition of f_m, and from rearranging. The Beta function with parameters M' and M'' is given by $\mathrm{Beta}(M', M'') = \int_0^1 v^{M'-1}(1-v)^{M''-1}\mathrm{d}v$. The fourth equality uses this function to rewrite the integral expression. The fifth equality follows from using that for integer parameters M' and M'', $\mathrm{Beta}(M', M'') = (M'-1)!(M''-1)!/(M'+M''-1)!$. The sixth equality follows from pulling $(M-1)!^2/(2M-1)!$ out of the brackets and from using that $2M-1 = N$. The seventh equality follows from simplifying.

By using (B.4) in (B.3), I obtain after rewriting the denominator

$$V^{*|\mathrm{m}(X)} = -\sigma^2 + \frac{1}{4} \cdot \frac{(N+1)^2}{(N+1)^2 - 1} \cdot \bar{t}^2. \qquad \text{q.e.d.}$$

Proof of Proposition 1

I prove (b) before (a) and (c). Fix for the entire proof any mechanism (ϕ_a, R) with $R \in \mathcal{R}$.

(b) Consider agent i's reporting problem when the other agents report according to $\widehat{r}_{-i} = (\widehat{r}_j, \ldots, \widehat{r}_j)$. By the discussion in the text, choosing $r_i \in R$ to maximize $U_i(x_i, r_i, \widehat{r}_{-i}(X_{-i}))$ is equivalent to choosing $r_i \in R$ to minimize $\mathbf{E}_X[\phi_a(r_i, \widehat{r}_{-i}(X_{-i})) - \theta(X_i, X_{-i})|X_i = x_i]^2$. By using the definition of ϕ_a and θ, the objective function can be written as $1/N^2 \cdot (r_i - \widehat{r}_i^a(x_i, b))^2$ with $b = \mathbf{E}_{X_j}[\widehat{r}_j(X_j)]$. Since boundedness of R implies that $\mathbf{E}_{X_j}[\widehat{r}_j(X_j)]$ exists, this problem is well–defined. Because R is closed, a report which lies closest to $\widehat{r}_i^a(x_i, b)$ does exist. It follows that $\widehat{r}_i$ is optimal for agent i if and only if $\widehat{r}_i(x_i)$ is for any x_i a report which lies closest to $\widehat{r}_i^a(x_i, b)$. Hence, $\widehat{r}_i$ constitutes a sBNE if and only if $\widehat{r}_i(x_i)$ selects for any x_i a report which lies closest to $\widehat{r}_i^a(x_i, b)$ with $b = \mathbf{E}_{X_i}[\widehat{r}_i(X_i)]$. This is (b).

(a) and (c) For any $b \in \mathbb{R}$, let $\widehat{r}_i(x_i, b) : \mathcal{X} \times \mathbb{R} \to R$ be a function which selects for any x_i an admissible report which lies closest to $N(1-\alpha)x_i - (N-1)b$. To prove existence of a sBNE, I have to show that there exists a $b \in \mathbb{R}$ such that $\mathbf{E}_{X_i}[\widehat{r}_i(X_i, b)] = b$. The left–hand side, $\mathbf{E}_{X_i}[\widehat{r}_i(X_i, b)]$, decreases continuously in b with $\lim_{b\to-\infty} \mathbf{E}_X[\widehat{r}_i(X_i, b)] = \max R$ and $\lim_{b\to+\infty} \mathbf{E}_X[\widehat{r}_i(X_i, b)] = \min R$. The right–hand side, b, increases continuously and strictly with $\lim_{b\to-\infty} b = -\infty < \max R$ and $\lim_{b\to+\infty} b = +\infty > \min R$. It follows by an Intermediate Value Theorem that there exists a $b \in \mathbb{R}$ such that $\mathbf{E}_{X_i}[\widehat{r}_i(X_i, b)] = b$. This is (a). Since the left–hand side of $\mathbf{E}_{X_i}[\widehat{r}_i(X_i, b)] = b$ decreases and the right–hand side increases strictly, there exists a unique $b \in \mathbb{R}$ for which the equation holds. This is the first part of (c). It follows that two sBNE can differ only for attributes x_i for which two closest admissible reports exist. Because this can only be the case for a set of attributes with measure zero, any equilibrium of the considered mechanism attains the same welfare. This is the second part of (c). q.e.d.

Proof of Corollary 1

(a) By construction, $\widehat{r}_i$ selects for any x_i the unique closest report to $\widehat{r}_i^a(x_i, b^{a,I}(t', t''))$. It follows from Proposition 1 (b) that $\widehat{r}_i$ constitutes a sBNE if $\mathbf{E}_{X_i}[\widehat{r}_i(X_i)] = b^{a,I}(t', t'')$. It is straightforward to check that this is indeed the case. By Proposition 1 (c) and the existence of a unique closest report for any x_i, the sBNE is unique.

(b) Suppose $\widehat{r}_i$ constitutes a non–constant sBNE of (ϕ_a, R) with $R \in \mathcal{R}_I$. Because $\widehat{r}_i$ is non–constant and R is an interval, there must by Proposition 1 (b) exist $t', t'' \in \mathcal{X}$ with $t' < t''$ and $b \in \mathbb{R}$ such that $\widehat{r}_i(x_i) = \widehat{r}_i^a(t', b)$ if $x_i < t'$, $\widehat{r}_i(x_i) = \widehat{r}_i^a(x_i, b)$ if $x_i \in [t', t'']$ and $\widehat{r}_i(x_i) = \widehat{r}_i^a(t'', b)$ if $x_i > t''$. From this, I obtain

$$\begin{aligned}
\mathbf{E}_{X_i}[\widehat{r}_i(X_i)] &= F(t')\widehat{r}_i^a(t', b) + \int_{t'}^{t''} \widehat{r}_i^a(x_i, b) f(x_i) \mathrm{d}x_i + (1 - F(t'))\widehat{r}_i^a(t'', b) \\
&= N(1-\alpha)[F(t')t' + \int_{t'}^{t''} \widehat{r}_i^a(x_i, b) f(x_i) \mathrm{d}x_i + (1 - F(t''))t''] - (N-1)b \\
&= N b^{a,I}(t', t'') - (N-1)b.
\end{aligned}$$

Since $\widehat{r}_i$ being a sBNE requires by Proposition 1 (b) that $\mathbf{E}_{X_i}[\widehat{r}_i(X_i)] = b$, it follows that $b = b^{a,I}(t', t'')$ must be true. $\widehat{r}_i$ corresponds thus to the reporting behavior which is by Part (a) induced by the mechanism $(\phi_a, R^{a,I}(t', t''))$. q.e.d.

Proof of Corollary 2

(a) By Proposition 1 (b), I have only to show that an agent with attribute $x_i = t$ is indifferent between the two admissible reports. That is, $\widehat{r}_i^a(t, b) = (\underline{r}^{a,B}(t, \delta) + \overline{r}^{a,B}(t, \delta))/2$ with $b =$

$\mathbf{E}_{X_i}[\widehat{r}_i(X_i)]$. I have

$$\begin{aligned}\widehat{r}_i^a(t,\mathbf{E}_{X_i}[\widehat{r}_i(X_i)]) &= N(1-\alpha)t-(N-1)[F(t)\underline{r}^{a,B}(t,\delta)+(1-F(t))\overline{r}^{a,B}(t,\delta)] \\ &= N(1-\alpha)t-(N-1)[(1-\alpha)t-F(t)k(t)\delta+(1-F(t))\delta] \\ &= (1-\alpha)t+[(N-1)F(t)k(t)-(N-1)(1-F(t))]\delta\end{aligned}$$

and $(\underline{r}^{a,B}(t,\delta)+\overline{r}^{a,B}(t,\delta))/2=(1-\alpha)t+[1/2-k(t)/2]\delta$. It suffices thus to show that $(N-1)F(t)k(t)-(N-1)(1-F(t))=1/2-k(t)/2$. Since solving this equation for $k(t)$ gives me just how $k(t)$ is defined in the corollary, I am done.

(b) Suppose $\widehat{r}_i$ constitutes a non–constant sBNE of $(\phi_a,\{\underline{r},\overline{r}\})$ with $\underline{r}<\overline{r}$. Because $\widehat{r}_i$ is non–constant, there must by Proposition 1 (b) exist a threshold t such that $\widehat{r}_i^a(t,b)=(\underline{r}+\overline{r})/2$ with $b=F(t)\underline{r}+(1-F(t))\overline{r}$. It is always possible to choose $\delta\geq 0$ such that $\overline{r}=\overline{r}^{a,B}(t,\delta)$. When I use this in the indifference condition, I obtain

$$\begin{aligned}& N(1-\alpha)t-(N-1)[F(t)\underline{r}+(1-F(t))\overline{r}^{a,B}(t,\delta)]=(\underline{r}+\overline{r}^{a,B}(t,\delta))/2 \\ \Leftrightarrow\quad & [(N-1)F(t)+1/2]\underline{r}=N(1-\alpha)t-[(N-1)(1-F(t))+1/2]\overline{r}^{a,B}(t,\delta).\end{aligned}$$

Since using $\overline{r}^{a,B}(t,\delta)=(1-\alpha)t+\delta$ and solving then the equation for $\underline{r}$ gives me just $\underline{r}=\underline{r}^{a,B}(t,\delta)$, I have $\{\underline{r},\overline{r}\}=R^{a,B}(t,\delta)$. This proves that there exist t and δ such that $\widehat{r}_i$ constitutes an equilibrium of $(\phi_a,R^{a,B}(t,\delta))$. q.e.d.

Proof of Proposition 2

Fix for the entire proof any mechanism (ϕ_m,R) with $R\in\mathcal{R}$.

(a) I denote the strategy profile where every agent uses the strategy $\widehat{r}_i^{E1}$ (resp. $\widehat{r}_i^{E2}$) by $\widehat{r}^{E1}$ (resp. $\widehat{r}^{E2}$). Moreover, I analogously define the strategy profile $\widehat{r}_{-i}^{E1}$ (resp. $\widehat{r}_{-i}^{E2}$) which consists of strategies for all agents but agent i.

Suppose $\widehat{r}_i^{E1}$ constitutes a sBNE which is not a sBNE*. There must then exist a $x_i\in\mathcal{X}$ such that $\widehat{r}_i^{E1}(x_i)\notin \mathrm{cl}(R^*(\widehat{r}_i^{E1}))$. Define $\mathcal{X}_1:=\{x_i\in\mathcal{X}|\widehat{r}_i^{E1}(x_i)<\min(\mathrm{cl}(R^*(\widehat{r}_i^{E1})))\}$ and $\mathcal{X}_2:=\{x_i\in\mathcal{X}|\widehat{r}_i^{E1}(x_i)>\max(\mathrm{cl}(R^*(\widehat{r}_i^{E1})))\}$. By construction, $\mathcal{X}_1$ and $\mathcal{X}_2$ have measure zero. Define $\widehat{r}_i^{E2}:\mathcal{X}\to R$ by $\widehat{r}_i^{E2}(x_i)=\min(\mathrm{cl}(R^*(\widehat{r}_i^{E1})))$ if $x_i\in\mathcal{X}_1$, $\widehat{r}_i^{E2}(x_i)=\max(\mathrm{cl}(R^*(\widehat{r}_i^{E1})))$ if $x_i\in\mathcal{X}_2$ and $\widehat{r}_i^{E2}(x_i)=\widehat{r}_i^{E1}(x_i)$ else. Because $\mathrm{Prob}_X[\mathrm{m}(r_i,\widehat{r}_{-i}^{E2}(X_{-i}))=\mathrm{m}(r_i,\widehat{r}_{-i}^{E1}(X_{-i}))]=1$ for any $r_i\in R$, $U_i(x_i,r_i,\widehat{r}_{-i}^{E1})=U_i(x_i,r_i,\widehat{r}_{-i}^{E2})$ for any $r_i\in R$. It follows from the supposition that $\widehat{r}_i^{E1}(x_i)$ is for any x_i a maximizer of $U_i(x_i,r_i,\widehat{r}_{-i}^{E2})$. Moreover, since $\mathrm{m}(\widehat{r}_i^{E1}(x_i),\widehat{r}_{-i}^{E2}(x_{-i}))=\mathrm{m}(\widehat{r}_i^{E2}(x_i),\widehat{r}_{-i}^{E2}(x_{-i}))$, also $\widehat{r}_i^{E2}(x_i)$ is for any x_i a maximizer of $U_i(x_i,r_i,\widehat{r}_{-i}^{E2})$. That is, $\widehat{r}_i^{E2}$ constitutes a sBNE. Since $\mathrm{cl}(\widehat{r}_i^{E2}(\mathcal{X}))=\mathrm{cl}(R^*(\widehat{r}_i))$, $\widehat{r}_i^{E2}$ constitutes a sBNE*. Because $\mathrm{Prob}_X[\mathrm{m}(\widehat{r}^{E2}(X))=\mathrm{m}(\widehat{r}^{E1}(X))]=1$, $V(\phi_a\circ\widehat{r}^{E1})=V(\phi_a\circ\widehat{r}^{E2})$.

(b) It is straightforward that $\widehat{r}_i(x_i)=r$ constitutes for any $r\in R$ a sBNE*.

(c,"$\Rightarrow$") Suppose $\widehat{r}_i$ constitutes a sBNE*. Since $\mathrm{cl}(\widehat{r}_i(\mathcal{X}))=\mathrm{cl}(R^*(\widehat{r}_i))$ follows directly from the supposition, I need only to show that there exists $R_\subseteq\in\mathcal{R}_\subseteq(R)$ such that $\widehat{r}_i$ selects for any x_i a report from $R_\subseteq$ which lies closest to $\widehat{r}_i^m(x_i)$. Consider in the subsequent steps agent i's problem to choose a report from R when the other agents report according to $\widehat{r}_{-i}=(\widehat{r}_i,\ldots,\widehat{r}_i)$.

Step 1: $\widehat{r}_i$ is weakly increasing. Assume to the contrary that there exist $x_i',x_i''\in\mathcal{X}$ with $x_i'<x_i''$ such that $\widehat{r}_i(x_i')>\widehat{r}_i(x_i'')$. Incentive compatibility requires that $U_i(x_i',\widehat{r}_i(x_i'),\widehat{r}_{-i})-U_i(x_i',\widehat{r}_i(x_i''),\widehat{r}_{-i})\geq 0$ and that $U_i(x_i'',\widehat{r}_i(x_i''),\widehat{r}_{-i})-U_i(x_i'',\widehat{r}_i(x_i'),\widehat{r}_{-i})\geq 0$. Necessary for the two inequalities to hold simultaneously is $U_i(x_i',\widehat{r}_i(x_i'),\widehat{r}_{-i})-U_i(x_i'',\widehat{r}_i(x_i'),\widehat{r}_{-i})+U_i(x_i'',\widehat{r}_i(x_i''),\widehat{r}_{-i})-$

$U_i(x_i', \widehat{r}_i(x_i''), \widehat{r}_{-i}) \geq 0$. By using the definition of $U_i(x_i, r_i, \widehat{r}_{-i})$, multiplying out and simplifying, the inequality becomes $2(1-\alpha)(x_i'' - x_i')\mathbf{E}_X[\mathrm{m}(\widehat{r}_i(x_i''), \widehat{r}_{-i}(X_{-i})) - \mathrm{m}(\widehat{r}_i(x_i'), \widehat{r}_{-i}(X_{-i}))] \geq 0$. Because $x_i'' - x_i' > 0$ and $\mathrm{m}(\widehat{r}_i(x_i''), \widehat{r}_{-i}(X_{-i})) - \mathrm{m}(\widehat{r}_i(x_i'), \widehat{r}_{-i}(X_{-i})) \leq 0$ by the supposition, the inequality can only hold if $\mathbf{E}_X[\mathrm{m}(\widehat{r}_i(x_i''), \widehat{r}_{-i}(X_{-i})) - \mathrm{m}(\widehat{r}_i(x_i'), \widehat{r}_{-i}(X_{-i}))] = 0$. Since $\mathrm{cl}(\widehat{r}_i(\mathcal{X})) = \mathrm{cl}(R^*(\widehat{r}_i))$ implies that for any $r', r'' \in R$ with $r' \neq r''$ I have $\mathrm{m}(\widehat{r}_i(x_i''), \widehat{r}_{-i}(X_{-i})) \neq \mathrm{m}(\widehat{r}_i(x_i'), \widehat{r}_{-i}(X_{-i}))$ with positive probability, I obtain a contradiction. Hence, any sBNE* must be weakly increasing.

Step 2: There exists a set $R_\subseteq \in \mathcal{R}_\subseteq(R)$ such that $\widehat{r}_i$ selects for any x_i' a report from $R_\subseteq$ which lies closest to $\widehat{r}_i^m(x_i')$. Step 1 implies that $\widehat{r}_i$ selects only reports from $[\widehat{r}_i(\underline{t}), \widehat{r}_i(\overline{t})] \cap R$. Define $R_\subseteq := [\widehat{r}_i(\underline{t}), \widehat{r}_i(\overline{t})] \cap R$ and note that $R_\subseteq \in \mathcal{R}_\subseteq(R)$. I will complete this proof by showing that $\widehat{r}_i$ selects for any x_i' a report from $R_\subseteq$ which lies closest to $\widehat{r}_i^m(x_i')$. I distinguish four cases: First, $\widehat{r}_i$ jumps at x_i'. Second, $\widehat{r}_i$ is continuous and strictly increasing in an open interval around x_i'. Third, $\widehat{r}_i$ is continuous but only strictly increasing on one side of x_i'. Fourth, $\widehat{r}_i$ is locally constant at x_i'.

Step 2a: If $\widehat{r}_i$ jumps at x_i' from r' to r'', then r' and r'' lie equally close to $\widehat{r}^m(x_i')$ and there is no report in $R_\subseteq$ which lies closer to $\widehat{r}^m(x_i')$. The supposition that $\widehat{r}_i$ is a sBNE* and the monotonicity of $\widehat{r}_i$ (see Step 1) imply that $\widehat{r}_i$ can only jump at interior points. That is, $x_i' \in (\underline{t}, \overline{t})$.

Suppose that $\widehat{r}_i$ is not right–continuous at x_i'. That is, $r' := \widehat{r}_i(x_i') < \lim_{x_i \downarrow x_i'} \widehat{r}_i(x_i) =: r''$. Incentive compatibility requires $U_i(x_i', r', \widehat{r}_{-i}) - U_i(x_i', r'', \widehat{r}_{-i}) \geq 0$ and $\lim_{x_i \downarrow x_i'}[U_i(x_i, \widehat{r}_i(x_i), \widehat{r}_{-i}) - U_i(x_i, r', \widehat{r}_{-i})] \geq 0$. Since $U_i(x_i, r_i, \widehat{r}_{-i})$ is continuous in x_i and r_i, necessary for both inequalities to hold simultaneously is that $U_i(x_i', r', \widehat{r}_{-i}) - U_i(x_i', r'', \widehat{r}_{-i}) = 0$. I have

$$\begin{aligned}
& U_i(x_i', r', \widehat{r}_{-i}) - U_i(x_i', r'', \widehat{r}_{-i}) \\
= \; & \mathbf{E}_X[-(\mathrm{m}(r', \widehat{r}_{-i}(X_{-i})) - \theta(X_i, X_{-i}))^2 + (\mathrm{m}(r'', \widehat{r}_{-i}(X_{-i})) - \theta(X_i, X_{-i}))^2 | X_i = x_i'] \\
= \; & \mathbf{E}_X[(\mathrm{m}(r'', \widehat{r}_{-i}(X_{-i})) - \mathrm{m}(r', \widehat{r}_{-i}(X_{-i}))) \\
& \quad \cdot (\mathrm{m}(r'', \widehat{r}_{-i}(X_{-i})) + \mathrm{m}(r', \widehat{r}_{-i}(X_{-i})) - 2\theta(x_i', X_{-i}))] \\
= \; & \mathrm{Prob}_X[\mathrm{m}(x_i', X_{-i}) = x_i'] \cdot \mathbf{E}_X[(r'' - r')(r'' + r' - 2\theta(X_i, X_{-i}))|\mathrm{m}(x_i', X_{-i}) = x_i'] \\
= \; & \mathrm{Prob}_X[\mathrm{m}(x_i', X_{-i}) = x_i'] \cdot 2(r'' - r')((r'' + r')/2 - \widehat{r}_i^m(x_i')).
\end{aligned}$$

The transformations arise as follows: The first equality follows from using the structure of U_i. The second equality follows from multiplying out and simplifying. The third equality follows from using that $\mathrm{m}(r'', \widehat{r}_{-i}(X_{-i}))$ differs from $\mathrm{m}(r', \widehat{r}_{-i}(X_{-i}))$ if and only if $\widehat{r}_i(X_j) \leq r'$ for half of the other agents and $\widehat{r}_i(X_j) > r''$ for the other half of the other agents. The informational content revealed by this event is that $\mathrm{m}(x_i', X_{-i}) = x_i'$. The fourth equality follows from using the definition of $\widehat{r}_i^m$.

Note now first that $x_i \in (\underline{t}, \overline{t})$ implies $\mathrm{Prob}_X[\mathrm{m}(x_i', X_{-i}) = x_i'] > 0$. Second, note that $r'' > r'$ by the supposition of Step 2a. Both properties together imply that $U_i(x_i', r', \widehat{r}_{-i}) - U_i(x_i', r'', \widehat{r}_{-i}) = 0$ is equivalent to $(r'' + r')/2 = \widehat{r}_i^m(x_i')$. That is, r' and r'' must lie equally close to $\widehat{r}_i^m(x_i')$. Furthermore, it follows also from the above calculations that incentive compatibility would be violated if $R \cap (r', r'')$ was non–empty.

Since an analogous reasoning applies for the case in which $\widehat{r}_i$ is not left–continuous at x_i', I obtain the result in Step 2a.

Step 2b: If $\widehat{r}_i$ is continuous and strictly increasing at x_i', then $\widehat{r}_i(x_i') = \widehat{r}_i^m(x_i')$. Suppose there exists an interval $(x_i' - \epsilon, x_i' + \epsilon) \subset \mathcal{X}$ with $\epsilon > 0$ such that $\widehat{r}_i$ is continuous and strictly increasing on this interval. Assume to the contrary that $\widehat{r}_i(x_i') < \widehat{r}_i^m(x_i')$. For any $\epsilon' \in (0, \epsilon)$, I have then

$$\begin{aligned} & U_i(x_i', \widehat{r}_i(x_i'+\epsilon'), \widehat{r}_{-i}) - U_i(x_i', \widehat{r}_i(x_i'), \widehat{r}_{-i}) \\ = \; & \mathbf{E}_X[(\widehat{r}_i(\mathrm{m}(x_i'+\epsilon', X_{-i})) - \widehat{r}_i(\mathrm{m}(x_i', X_{-i}))) \\ & \cdot(2\theta(X_i, X_{-i}) - \widehat{r}_i(\mathrm{m}(x_i'+\epsilon', X_{-i})) - \widehat{r}_i(\mathrm{m}(x_i', X_{-i})))|X_i = x_i']. \end{aligned} \tag{B.5}$$

The equality follows from using the structure of U_i, multiplying out and simplifying. Necessary for $\widehat{r}_i(\mathrm{m}(x_i'+\epsilon', X_{-i})) - \widehat{r}_i(\mathrm{m}(x_i', X_{-i})) \neq 0$ is $\mathrm{m}(x_i', X_{-i}) \in [x_i', x_i'+\epsilon']$. I can partition the event that $\mathrm{m}(x_i', X_{-i}) \in [x_i', x_i'+\epsilon']$ into two subevents. In the first, say event $E1$, I additionally have $\forall j \neq i : X_j \notin [x_i', x_i'+\epsilon']$. This implies that $\widehat{r}_i(\mathrm{m}(x_i'+\epsilon', X_{-i})) = \widehat{r}_i(x_i'+\epsilon')$ and $\widehat{r}_i(\mathrm{m}(x_i', X_{-i})) = \widehat{r}_i(x_i')$. In the second event, say event $E2$, I additionally have $\exists j \neq i : X_j \in [x_i', x_i'+\epsilon']$. This implies $\widehat{r}_i(\mathrm{m}(x_i'+\epsilon', X_{-i})) - \widehat{r}_i(\mathrm{m}(x_i', X_{-i})) \leq \widehat{r}_i(x_i'+\epsilon') - \widehat{r}_i(x_i')$. By using this in (B.5) and by dividing both sides of the equation by $\widehat{r}_i(x_i'+\epsilon') - \widehat{r}_i(x_i')$, I obtain

$$\begin{aligned} & \frac{U_i(x_i', \widehat{r}_i(x_i'+\epsilon'), \widehat{r}_{-i}) - U_i(x_i', \widehat{r}_i(x_i'), \widehat{r}_{-i})}{\widehat{r}_i(x_i'+\epsilon') - \widehat{r}_i(x_i')} \\ = \; & \mathrm{Prob}_X[E_1] \cdot \mathbf{E}_X[2\theta(X_i, X_{-i}) - \widehat{r}_i(x_i'+\epsilon') - \widehat{r}_i(x_i')|X_i = x_i', E_1] \\ & + \mathrm{Prob}_X[E_2] \cdot \mathbf{E}_X[\frac{\widehat{r}_i(\mathrm{m}(x_i'+\epsilon', X_{-i})) - \widehat{r}_i(\mathrm{m}(x_i', X_{-i}))}{\widehat{r}_i(x_i'+\epsilon') - \widehat{r}_i(x_i')} \\ & \quad \cdot(2\theta(X_i, X_{-i}) - \widehat{r}_i(\mathrm{m}(x_i'+\epsilon', X_{-i})) - \widehat{r}_i(\mathrm{m}(x_i', X_{-i})))|X_i = x_i', E_2]. \end{aligned} \tag{B.6}$$

To prove that agent i has an incentive to deviate when his private signal is x_i', it suffices to argue that the right–hand side of (B.6) is strictly positive for some $\epsilon' \in (0, \epsilon)$. Sufficient for this is that the right–hand side converges to a strictly positive value as $\epsilon' \downarrow 0$. This property follows from four observations. First, $\lim_{\epsilon'\downarrow 0} \mathrm{Prob}_X[E1] = \mathrm{Prob}_X[\mathrm{m}(x_i', X_{-i}) = x_i']$. Second, $\lim_{\epsilon'\downarrow 0} \mathrm{Prob}_X[E2] = 0$. Third, $\mathbf{E}_X[\theta(X_i, X_{-i})|X_i = x_i', E_1] = (1-\alpha)x_i' + \alpha(\mathbf{E}_{X_j}[X_j|X_j < x_i'] + \mathbf{E}_{X_j}[X_j > x_i'+\epsilon'])/2$ such that $\lim_{\epsilon'\downarrow 0} \mathbf{E}_X[\theta(X_i, X_{-i})|X_i = x_i', E_1] = \widehat{r}_i^m(x_i')$. By using this and continuity of $\widehat{r}_i$ at x_i', I obtain that the first expected value expression in (B.6) converges to $2(\widehat{r}_i^m(x_i') - \widehat{r}_i(x_i'))$ as $\epsilon' \downarrow 0$. Fourth, conditional on the event E_2, the fraction term in the second expected value expression in (B.6) assumes values in $[0, 1]$. This implies that the second expected value expression is bounded. The four observations together imply that the right–hand side of (B.6) converges towards $\mathrm{Prob}_X[\mathrm{m}(x_i', X_{-i}) = x_i']2(\widehat{r}_i^m(x_i') - \widehat{r}_i(x_i'))$. Since x_i' is interior, the probability expression is strictly positive. Since $\widehat{r}_i(x_i') < \widehat{r}_i^m(x_i')$ by my supposition, I obtain a contradiction. Since a contradiction to $\widehat{r}_i(x_i') > \widehat{r}_i^m(x_i')$ can be derived analogously, I obtain the result in Step 2b.

Step 2c: If $\widehat{r}_i$ is continuous and one–sided strictly increasing at x_i', then $\widehat{r}_i(x_i') = \widehat{r}_i^m(x_i')$. Suppose without loss of generality that there exists an interval $[x_i', x_i'+\epsilon)$ with $\epsilon > 0$ such that $\widehat{r}_i$ is continuous and strictly increasing on this interval. Since Step 2b applies to any $x_i \in (x_i', x_i'+\epsilon)$, there exists a sequence of points x_i which converges towards x_i' and for which Step 2b applies. The result follows then from continuity of $\widehat{r}_i^m$ and the supposed continuity of $\widehat{r}_i$ at x_i'.

Step 2d: If $\widehat{r}_i$ is constant at x_i', then $\widehat{r}_i(x_i')$ is the report from $R_\subseteq$ which lies closest to $\widehat{r}_i^m(x_i')$. Suppose $\widehat{r}_i$ is locally constant at x_i' with $\widehat{r}_i(x_i') = r$. I explain why no smaller report in $R_\subseteq$ exists which lies closer to $\widehat{r}_i^m(x_i')$ than r. The argument for why no larger report with this property exists is then analogous. Let $x_i'' = \inf\{x_i \in \mathcal{X}|\widehat{r}_i(x_i) = r\}$. If $x_i'' = \underline{t}$, then $r = \min(R_\subseteq)$ by the construction of the set $R_\subseteq$. That is, there exists no smaller report in $R_\subseteq$. If $x_i'' > \underline{t}$, then either Step 2a or 2c applies to x_i''. By these steps, there exists no report in $R_\subseteq$ which is smaller than r and which lies closer to $\widehat{r}_i^m(x_i'')$. Since $\widehat{r}_i^m$ is strictly increasing, it follows that there is also no report in $R_\subseteq$ which is smaller than r and which lies closer to $\widehat{r}_i^m(x_i')$.

(c,"$\Leftarrow$") Suppose there exists a set $R_{\subseteq} \in \mathcal{R}_{\subseteq}(R)$ such that $\widehat{r}_i$ selects for any x_i a report from $R_{\subseteq}$ which lies closest to $\widehat{r}_i^m(x_i)$ and suppose that $\mathrm{cl}(\widehat{r}_i(\mathcal{X})) = \mathrm{cl}(R^*(\widehat{r}_i))$. Let $\widehat{r}_{-i} = (\widehat{r}_i, \ldots, \widehat{r}_i)$. The second supposition implies that $\widehat{r}_i$ is a sBNE* if it is a sBNE. It suffices thus to show that $\widehat{r}_i(x_i') \in \arg\max_{r_i \in R} U_i(x_i', r_i, \widehat{r}_{-i})$ for any $x_i' \in \mathcal{X}$. Since $\widehat{r}_i^m$ is strictly increasing, the first supposition implies that $\widehat{r}_i$ is weakly increasing. As a consequence, reports outside $[\widehat{r}_i(\underline{t}), \widehat{r}_i(\overline{t})]$ have no chance of being selected as collective decision. Agent i is thus indifferent between choosing any of those reports and choosing either $\widehat{r}_i(\underline{t})$ or $\widehat{r}_i(\overline{t})$. I need thus only to show that $\widehat{r}_i(x_i') \in \arg\max_{r_i \in [\widehat{r}_i(\underline{t}), \widehat{r}_i(\overline{t})] \cap R} U_i(x_i', r_i, \widehat{r}_{-i})$ for any $x_i' \in \mathcal{X}$. Since the first supposition and continuity of $\widehat{r}_i^m$ imply that $\widehat{r}_i(\mathcal{X}) = [\widehat{r}_i(\underline{t}), \widehat{r}_i(\overline{t})] \cap R_{\subseteq}$ and since $R_{\subseteq} \in \mathcal{R}_{\subseteq}(R)$ implies that $[\widehat{r}_i(\underline{t}), \widehat{r}_i(\overline{t})] \cap R_{\subseteq} = [\widehat{r}_i(\underline{t}), \widehat{r}_i(\overline{t})] \cap R$, I only have to show that $\widehat{r}_i(x_i') \in \arg\max_{x_i'' \in \mathcal{X}} U_i(x_i', \widehat{r}_i(x_i''), \widehat{r}_{-i})$ for any $x_i' \in \mathcal{X}$. I can prove this by showing that for any $x_i' \in \mathcal{X}$, $U_i(x_i', \widehat{r}_i(x_i''), \widehat{r}_{-i})$ is weakly increasing in x_i'' on $[\underline{t}, x_i')$ and weakly decreasing in x_i'' on $(x_i', \overline{t}]$. I will show the first property, the second property follows analogously. I can then conclude that $\widehat{r}_i(x_i')$ is the only candidate for equilibrium reporting behavior. Since an equilibrium exists, it must thus be specified by $\widehat{r}_i$.

Consider thus any $x_i'' < x_i'$. Suppose $\epsilon \in (0, x_i' - x_i'')$. It suffices for me to show that $U_i(x_i', \widehat{r}_i(x_i'' + \epsilon), \widehat{r}_{-i}) - U_i(x_i', \widehat{r}_i(x_i''), \widehat{r}_{-i}) \geq 0$ for any sufficiently small ϵ. Since it follows from $\widehat{r}_i$ being weakly increasing that this holds trivially if there exists $\epsilon \in (0, x_i' - x_i'')$ such that $\widehat{r}_i(x_i'' + \epsilon) - \widehat{r}_i(x_i'') > 0$, suppose $\widehat{r}_i(x_i'' + \epsilon) - \widehat{r}_i(x_i'') > 0$ for all $\epsilon \in (0, x_i' - x_i'')$. Denote the event in which $\mathrm{m}(x_i'', X_{-i}) \in [x_i'', x_i'' + \epsilon]$ and $\forall j \neq i : X_j \notin [x_i'', x_i'' + \epsilon]$ by E_1 and the event in which $\mathrm{m}(x_i'', X_{-i}) \in [x_i'', x_i'' + \epsilon]$ and $\exists j \neq i : X_j \in [x_i'', x_i'' + \epsilon]$ by E_2. By a derivation which is analogous to that of (B.6), I obtain then

$$
\begin{aligned}
& \frac{U_i(x_i', \widehat{r}_i(x_i'' + \epsilon), \widehat{r}_{-i}) - U_i(x_i', \widehat{r}_i(x_i''), \widehat{r}_{-i})}{\widehat{r}_i(x_i'' + \epsilon) - \widehat{r}_i(x_i'')} \\
= \ & \mathrm{Prob}_X[E_1] \cdot \mathbf{E}_X[2\theta(x_i', X_{-i}) - \widehat{r}_i(x_i'' + \epsilon) - \widehat{r}_i(x_i'') | E_1] \\
& + \mathrm{Prob}_X[E_2] \cdot \mathbf{E}_X[\frac{\widehat{r}_i(\mathrm{m}(x_i'' + \epsilon, X_{-i})) - \widehat{r}_i(\mathrm{m}(x_i'', X_{-i}))}{\widehat{r}_i(x_i'' + \epsilon) - \widehat{r}_i(x_i)} \\
& \quad \cdot (2\theta(x_i', X_{-i}) - \widehat{r}_i(\mathrm{m}(x_i'' + \epsilon, X_{-i})) - \widehat{r}_i(\mathrm{m}(x_i'', X_{-i}))) | E_2].
\end{aligned}
\tag{B.7}
$$

Note now that $\lim_{\epsilon \downarrow 0} \mathrm{Prob}_X[E1] = \mathrm{Prob}_X[\mathrm{m}(x_i'', X_{-i}) = x_i'']$, $\lim_{\epsilon \downarrow 0} \mathrm{Prob}_X[E2] = 0$ and that the second expected value expression is bounded since the fraction term lies always in $[0, 1]$ conditional on the event $E2$. These three observations together imply that

$$
\begin{aligned}
& \lim_{\epsilon \downarrow 0} \frac{U_i(x_i', \widehat{r}_i(x_i'' + \epsilon), \widehat{r}_{-i}) - U_i(x_i', \widehat{r}_i(x_i''), \widehat{r}_{-i})}{\widehat{r}_i(x_i'' + \epsilon) - \widehat{r}_i(x_i'')} \\
= \ & \mathrm{Prob}_X[\mathrm{m}(x_i'', X_{-i}) = x_i''] \cdot \lim_{\epsilon \downarrow 0} \mathbf{E}_X[2\theta(x_i', X_{-i}) - \widehat{r}_i(x_i'' + \epsilon) - \widehat{r}_i(x_i'') | E1].
\end{aligned}
$$

When I use that $\theta(x_i', X_{-i}) = (1 - \alpha)(x_i' - x_i'') + \theta(x_i'', X_{-i})$ and that $\lim_{\epsilon \downarrow 0} \mathbf{E}_X[\theta(x_i'', X_{-i}) | E1] = \widehat{r}_i^m(x_i'')$, I obtain

$$
\begin{aligned}
& \lim_{\epsilon \downarrow 0} \frac{U_i(x_i', \widehat{r}_i(x_i'' + \epsilon), \widehat{r}_{-i}) - U_i(x_i', \widehat{r}_i(x_i''), \widehat{r}_{-i})}{\widehat{r}_i(x_i'' + \epsilon) - \widehat{r}_i(x_i'')} \\
= \ & \mathrm{Prob}_X[\mathrm{m}(x_i'', X_{-i}) = x_i''] \cdot (2(1 - \alpha)(x_i' - x_i'') + 2\widehat{r}_i^m(x_i'') - \lim_{\epsilon \downarrow 0} \widehat{r}_i(x_i'' + \epsilon) - \widehat{r}_i(x_i'')).
\end{aligned}
$$

I need to distinguish two cases. First, suppose $\lim_{\epsilon \downarrow 0} \widehat{r}_i(x_i'' + \epsilon) > \widehat{r}_i(x_i'')$. $\widehat{r}_i$ must then jump upwards at x_i'' between two reports which lie equally close to $\widehat{r}_i^m(x_i'')$. This implies $\lim_{\epsilon \downarrow 0} \widehat{r}_i(x_i'' + \epsilon) + \widehat{r}_i(x_i'') = 2\widehat{r}_i^m(x_i'')$. Second, suppose $\lim_{\epsilon \downarrow 0} \widehat{r}_i(x_i'' + \epsilon) = \widehat{r}_i(x_i'')$. By the supposition that $\widehat{r}_i$

is locally strictly increasing at x_i'', $\widehat{r}_i(x_i'') = \widehat{r}_i^m(x_i'')$ must be true. I obtain again $\lim_{\epsilon\downarrow 0}\widehat{r}_i(x_i''+\epsilon)+\widehat{r}_i(x_i'') = 2\widehat{r}_i^m(x_i'')$. Hence, in either case,

$$\begin{aligned} & \lim_{\epsilon\downarrow 0}\frac{U_i(x_i',\widehat{r}_i(x_i''+\epsilon),\widehat{r}_{-i}) - U_i(x_i',\widehat{r}_i(x_i''),\widehat{r}_{-i})}{\widehat{r}_i(x_i''+\epsilon)-\widehat{r}_i(x_i'')} \\ = \;& \mathrm{Prob}_X[\mathrm{m}(x_i'',X_{-i}) = x_i'']\cdot 2(1-\alpha)(x_i'-x_i''). \end{aligned}$$

Since my supposition that $x_i' > x_i''$ implies that this is non–negative, I can conclude that $U_i(x_i',\widehat{r}_i(x_i''),\widehat{r}_{-i})$ is weakly increasing in x_i'' on $[\underline{t},x_i')$.

(d) Two sBNE* with the same $R_\subseteq$ can differ only for signals x_i for which $R_\subseteq$ contains two distinct reports which are closest to $\widehat{r}^m(x_i)$. Since this can only be the case for a set of attributes with measure zero, welfare coincides. q.e.d.

Proof of Corollary 3

The corollary follows straightforwardly from Proposition 2. q.e.d.

Proof of Corollary 4

The corollary follows straightforwardly from Proposition 2. q.e.d.

Proof of Lemma 3.

(a) By using the definition of $\widehat{r}_i^m$ and $\widehat{r}_i^{m,*}$, I can write $\widehat{r}_i^m(x_i)-\widehat{r}_i^{m,*}(x_i) = ((N-1)/N-\alpha)\cdot(x_i-h(x_i))$. Since Lemma A1 (a) in Appendix A implies $\widehat{r}_i^m(\underline{t})-\widehat{r}_i^{m,*}(\underline{t}) = ((N-1)/N-\alpha)\cdot\underline{t}/2<0$ and $\widehat{r}_i^m(\overline{t})-\widehat{r}_i^{m,*}(\overline{t}) = ((N-1)/N-\alpha)\cdot\overline{t}/2>0$, I obtain the result.

(b) Suppose $F\in\mathcal{F}_h$. By Part (a) and continuity of $x_i-h(x_i)$, I can apply an Intermediate Value Theorem to obtain that $\widehat{r}_i^m$ and $\widehat{r}_i^{m,*}$ intersect at least once on $(\underline{t},\overline{t})$. Since Lemma A1 (b) in Appendix A implies that $x_i-h(x_i)$ is strictly increasing, there exists a unique point of intersection $x_i'\in(\underline{t},\overline{t})$. This together with Part (a) implies that $\widehat{r}_i^m(x_i)<\widehat{r}_i^{m,*}(x_i)$ if $x_i<x_i'$ and $\widehat{r}_i^m(x_i)>\widehat{r}_i^{m,*}(x_i)$ if $x_i>x_i'$.

(c) Suppose $F\in\mathcal{F}_h\cap\mathcal{F}_s$. Lemma A1 (a) in Appendix A implies for $F\in\mathcal{F}_s$ that $h(0)=0$. By using this in Part (b), I obtain that the unique point of intersection must be at zero. Hence, $|\widehat{r}_i^{m,*}(x_i)|\le|\widehat{r}_i^m(x_i)|$. Since $F\in\mathcal{F}_s$ implies $\widehat{r}_i^m(0)=0$ and since $F\in\mathcal{F}_h$ and Lemma A1 (b) in Appendix A imply that $\widehat{r}_i^m$ increases with a slope smaller than one, I further have $|\widehat{r}_i^m(x_i)|\le|x_i|$.

(d) Suppose $X_i\sim\mathrm{U}[-\overline{t},\overline{t}]$. By Lemma A1 (d) in Appendix A, $h(x_i)=x_i/2$. The formulas for uniformly distributed attributes follow from using this in the definition of $\widehat{r}_i^{m,*}$ and $\widehat{r}_i^m$.q.e.d.

Derivation of (4).

It follows from Proposition 2 (c) that any sBNE* $\widehat{r}_i$ is weakly increasing. For any weakly increasing reporting strategy $\widehat{r}_i$, I have

$$\begin{aligned} & V(\phi_m\circ(\widehat{r}_i,\ldots,\widehat{r}_i)) \\ = \;& N\cdot\mathbf{E}_X[-((\widehat{r}_i(\mathrm{m}(X))-\widehat{r}_i^{m,*}(\mathrm{m}(X)))+(\widehat{r}_i^{m,*}(\mathrm{m}(X))-\overline{X}))^2] \\ = \;& N\cdot\mathbf{E}_X[-(\widehat{r}_i(\mathrm{m}(X))-\widehat{r}_i^{m,*}(\mathrm{m}(X)))^2]+N\cdot\mathbf{E}_X[-(\widehat{r}_i^{m,*}(\mathrm{m}(X))-\overline{X})^2] \end{aligned}$$

$$
\begin{aligned}
& +N\cdot\mathbf{E}_X[-(\widehat{r}_i(\mathrm{m}(X))-\widehat{r}_i^{m,*}(\mathrm{m}(X)))\cdot(\widehat{r}_i^{m,*}(\mathrm{m}(X))-\mathbf{E}_X[\overline{X}|\mathrm{m}(X)])] \\
= \quad & N\cdot\mathbf{E}_X[-(\widehat{r}_i(\mathrm{m}(X))-\widehat{r}_i^{m,*}(\mathrm{m}(X)))^2]+V^{*|\mathrm{m}(X)}.
\end{aligned}
$$

The transformations arise as follows: The first equality follows from adding and subtracting $\widehat{r}_i^{m,*}(\mathrm{m}(X))$, and because monotonicity of $\widehat{r}_i$ allows me to write $\mathrm{m}(\widehat{r}_i(X_1),\ldots,\widehat{r}_i(X_N))=\widehat{r}_i(\mathrm{m}(X))$. The second equality follows from multiplying out and from applying then the Law of Iterated Expectations to the last expected value expression. The third equality follows from using that $V^{*|\mathrm{m}(X)}=N\cdot\mathbf{E}_X[-(\widehat{r}_i^{m,*}(\mathrm{m}(X))-\overline{X})^2]$ and that the last expected value expression is zero because $\widehat{r}_i^{m,*}(\mathrm{m}(X))=\mathbf{E}_X[\overline{X}|\mathrm{m}(X)]$. This is (4). q.e.d.

Derivation of (6)

For any reporting strategy $\widehat{r}_i$, I have

$$
\begin{aligned}
& V(\phi_a\circ(\widehat{r}_i,\ldots,\widehat{r}_i)) \\
= \quad & N\cdot\mathbf{E}_X[-(1/N\cdot\textstyle\sum_i(\widehat{r}_i(X_i)-X_i))^2] \\
= \quad & 1/N\cdot\mathbf{E}_X[-(\textstyle\sum_i(\widehat{r}_i(X_i)-X_i)^2+\sum_i\sum_{j\neq i}(\widehat{r}_i(X_i)-X_i)(\widehat{r}_j(X_j)-X_j))] \\
= \quad & \mathbf{E}_{X_i}[-(\widehat{r}_i(X_i)-X_i)^2]-(N-1)\mathbf{E}_{X_i}[\widehat{r}_i(X_i)-X_i]^2.
\end{aligned}
$$

The transformations arise as follows: The first equality follows from using the definition of $V(\widehat{y})$ and of ϕ_a. The second equality follows from multiplying out. The third equality follows from using that attributes are independently and identically distributed.

By using then that $\mathbf{E}_{X_i}[X_i]=0$ by the normalization of the distribution and that $\widehat{r}_i^{a,*}(X_i)=X_i$, I obtain

$$
V(\phi_a,R)=\mathbf{E}_{X_i}[-(\widehat{r}_i(X_i)-\widehat{r}_i^{a,*}(X_i))^2]-(N-1)b^2 \tag{B.8}
$$

with $b=\mathbf{E}_{X_i}[\widehat{r}_i(X_i)]$.

It remains to show that (6) corresponds to (B.8). I can rewrite (6) as follows:

$$
\begin{aligned}
& \mathbf{E}_{X_i}[-((\widehat{r}_i(X_i)-\widehat{r}_i^{a,*}(X_i))+(\sqrt{N}-1)b)^2] \\
= \quad & \mathbf{E}_{X_i}[-(\widehat{r}_i(X_i)-\widehat{r}_i^{a,*}(X_i))^2-2(\widehat{r}_i(X_i)-\widehat{r}_i^{a,*}(X_i))(\sqrt{N}-1)b-(\sqrt{N}-1)^2b^2] \\
= \quad & \mathbf{E}_{X_i}[-(\widehat{r}_i(X_i)-\widehat{r}_i^{a,*}(X_i))^2]-(2\sqrt{N}-2)b\mathbf{E}_{X_i}[\widehat{r}_i(X_i)-\widehat{r}_i^{a,*}(X_i)]-(N-2\sqrt{N}+1)b^2 \\
= \quad & \mathbf{E}_{X_i}[-(\widehat{r}_i(X_i)-\widehat{r}_i^{a,*}(X_i))^2]-(2\sqrt{N}-2)b^2-(N-2\sqrt{N}+1)b^2
\end{aligned}
$$

The first equation follows from multiplying out. The second equation follows from simplifying. The third equation follows from using that $\mathbf{E}_{X_i}[\widehat{r}_i(X_i)]=b$ and that $\mathbf{E}_{X_i}[\widehat{r}_i^{a,*}(X_i)]=0$. (B.8) follows then from simplifying. q.e.d.

Proof of Proposition 3

Step 1: An upper bound on welfare attained by the optimal binary average mechanism. By Corollary 2, the information reflected by any binary average mechanism is $\mathrm{sgn}(X-t)$ for some $t\in\mathcal{X}$. By Lemma 2 (b), an upper bound on welfare attained by any mechanism which reflects only this information is $V^{*|(\mathrm{sgn}(X_1-t),\ldots,\mathrm{sgn}(X_N-t))}=-\sigma^2+c(t)$ with $c(t)=F(t)\mathbf{E}_{X_i}[X_i|X_i\leq t]^2+(1-F(t))\mathbf{E}_{X_i}[X_i|X_i\geq t]^2$. An upper bound on welfare attained by the optimal binary average mechanism is thus $\max_{t\in\mathcal{X}}-\sigma^2+c(t)$. By writing the expected value expressions in the definition of $c(t)$ as integrals, I get $c(t) = (\int_{\underline{t}}^{t} x_i f(x_i)\mathrm{d}x_i)^2/F(t) + (\int_{t}^{\overline{t}} x_i f(x_i)\mathrm{d}x_i)^2/(1-F(t))$. It follows

$$\begin{aligned}
c'(t) &= \frac{2tf(t)\left(\int_{\underline{t}}^{t} x_i f(x_i)\mathrm{d}x_i\right)F(t) - f(t)\left(\int_{\underline{t}}^{t} x_i f(x_i)\mathrm{d}x_i\right)^2}{F(t)^2} \\
&\quad + \frac{-2tf(t)\left(\int_{t}^{\overline{t}} x_i f(x_i)\mathrm{d}x_i\right)(1-F(t)) + f(t)\left(\int_{t}^{\overline{t}} x_i f(x_i)\mathrm{d}x_i\right)^2}{(1-F(t))^2} \\
&= f(t)\cdot\big(2t\mathbf{E}_{X_i}[X_i|X_i \leq t] - \mathbf{E}_{X_i}[X_i|X_i \leq t]^2 \\
&\qquad -2t\mathbf{E}_{X_i}[X_i|X_i \geq t] + \mathbf{E}_{X_i}[X_i|X_i \geq t]^2\big) \\
&= 2f(t)\cdot(\mathbf{E}_{X_i}[X_i|X_i \geq t] - \mathbf{E}_{X_i}[X_i|X_i \leq t])\cdot(h(t)-t).
\end{aligned}$$

The transformations arise as follows: The second equality follows from writing the integrals as expected values again. The third equality follows from simplifying.

Since $F \in \mathcal{F}_s$ implies $h(0) = 0$ (see Lemma A1 (a) in Appendix A), $c(t)$ assumes a local extremum at $t = 0$. Because $\mathbf{E}_{X_i}[X_i|X_i \geq t] - \mathbf{E}_{X_i}[X_i|X_i \leq t] > 0$ and because $F \in \mathcal{F}_h$ implies that $h(t) - t$ is strictly decreasing (see Lemma A1 (b) in Appendix A), $c'(t)$ changes its sign only once from positive to negative. Hence, $c(t)$ assumes a local maximum at $t = 0$. An upper bound on welfare attained by the optimal binary average mechanism is thus given by $V^{*|(\mathrm{sgn}(X_1),\ldots,\mathrm{sgn}(X_N))} = -\sigma^2 + c(0)$.

Step 2: The optimal binary average mechanism. Consider the mechanism $(\phi_a, R^{a,B}(0,\delta))$ with $\delta = \mathbf{E}_{X_i}[X_i|X_i \geq 0]$. By Corollary 2 (a), each agent i chooses in any sBNE $\widehat{r}_i$ the report $\overline{r}^{a,B}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0]) = \mathbf{E}_{X_i}[X_i|X_i \geq 0]$ if $x_i > 0$ and the report $\underline{r}^{a,B}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0]) = -\mathbf{E}_{X_i}[X_i|X_i \geq 0]$ if $x_i < 0$. Since $F \in \mathcal{F}_s$ implies $-\mathbf{E}_{X_i}[X_i|X_i \geq 0] = \mathbf{E}_{X_i}[X_i|X_i \leq 0]$ and $F(0) = 1/2$, $\mathbf{E}_{X_i}[\widehat{r}_i(X_i)] = 0$. This implies that $V(\phi_a, R^{a,B}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0]))$ is given by (6) with $b = 0$ and the just described reporting behavior. By simplifying, I obtain $V(\phi_a, R^{a,B}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0])) = -\sigma^2 + c(0)$. By Step 1, the mechanism $(\phi_a, \{\mathbf{E}_{X_i}[X_i|X_i \leq 0], \mathbf{E}_{X_i}[X_i|X_i \geq 0]\})$ is optimal among all binary average mechansims.

Step 3: The optimal binary median mechanism is strictly worse than the optimal binary average mechanism. By Corollary 4, there exists for any binary median mechanism some $t \in \mathcal{X}$ such that the information reflected by this mechanism is $\mathrm{sgn}(\mathrm{m}(X) - t)$. By Lemma 1 (a), an upper bound on welfare attained by any mechanism which reflects the information $\mathrm{sgn}(\mathrm{m}(X) - t)$ is $V^{*|\mathrm{sgn}(\mathrm{m}(X)-t)}$. I have by the formula in this lemma

$$\begin{aligned}
V^{*|\mathrm{sgn}(\mathrm{m}(X)-t)} &= -\sigma^2 + N\cdot\mathbf{E}_X[\mathbf{E}_X[\overline{X}|\mathrm{sgn}(\mathrm{m}(X)-t)]^2] \\
&= -\sigma^2 + N\cdot\mathbf{E}_X[\mathbf{E}_X[\mathbf{E}_X[\overline{X}|\mathrm{sgn}(X-t)]|\mathrm{sgn}(\mathrm{m}(X)-t)]^2] \\
&< -\sigma^2 + N\cdot\mathbf{E}_X[\mathbf{E}_X[\mathbf{E}_X[\overline{X}|\mathrm{sgn}(X-t)]^2|\mathrm{sgn}(\mathrm{m}(X)-t)]] \\
&= -\sigma^2 + N\cdot\mathbf{E}_X[\mathbf{E}_X[\overline{X}|\mathrm{sgn}(X-t)]^2] \\
&= V^{*|(\mathrm{sgn}(X_1-t),\ldots,\mathrm{sgn}(X_N-t))} \\
&\leq V^{*|(\mathrm{sgn}(X_1),\ldots,\mathrm{sgn}(X_N))} \\
&= V(\phi_a, \{\mathbf{E}_{X_i}[X_i|X_i \leq 0], \mathbf{E}_{X_i}[X_i|X_i \geq 0]\}).
\end{aligned}$$

The transformations arise as follows: The second equality follows from the Law of Iterated Expectations because $\mathrm{sgn}(X - t)$ is a sufficient statistic for $\mathrm{sgn}(\mathrm{m}(X) - t)$. The first inequality follows because $\mathrm{Var}_X[\mathbf{E}_X[\overline{X}|\mathrm{sgn}(X - t)]|\mathrm{sgn}(\mathrm{m}(X) - t)] > 0$ and because I can rewrite this inequality as $\mathbf{E}_X[\mathbf{E}_X[\overline{X}|\text{sgn}(X-t)]|\text{sgn}(\text{m}(X)-t)]^2 < \mathbf{E}_X[\mathbf{E}_X[\overline{X}|\text{sgn}(X-t)]^2|\text{sgn}(\text{m}(X)-t)]$. The third equality follows from the Law of Iterated Expectations. The fourth equality follows from Lemma 1 (a). The second inequality follows from the analysis in Step 1. The fifth equality follows from Step 2. Since this reasoning applies for any $t \in \mathcal{X}$, I obtain that the optimal binary median mechanism must be strictly worse than the optimal binary average mechanism. q.e.d.

Proof of Lemma 4.

Fix any $\alpha \in [0, 1)$ and consider $N \geq 1/(1-\alpha)$.

Step 1: Welfare attained by a specific binary average mechanism. Consider the mechanism $(\phi_a, R^{a,B}(0, \delta))$ with $\delta = \mathbf{E}_{X_i}[X_i|X_i \geq 0]$. By Corollary 2 (a), each agent i chooses in any sBNE $\widehat{r}_i$ the report $\overline{r} := \overline{r}^{a,B}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0]) = \mathbf{E}_{X_i}[X_i|X_i \geq 0]$ if $x_i > 0$ and the report $\underline{r} := \underline{r}^{a,B}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0]) = -k(0)\mathbf{E}_{X_i}[X_i|X_i \geq 0]$ if $x_i < 0$. By the representation (B.8) of $V(\phi_a, R)$ which I obtain when I derive (6) in this Appendix,

$$\begin{aligned} V(\phi_a, R^{a,B}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0])) &= F(0)\mathbf{E}_{X_i}[-(\underline{r} - X_i)^2|X_i \leq 0] \\ &\quad +(1 - F(0))\mathbf{E}_{X_i}[-(\overline{r} - X_i)^2|X_i \geq 0] \\ &\quad -(N-1)(F(0)\underline{r} + (1 - F(0))\overline{r})^2. \end{aligned}$$

I can simplify the expected welfare loss which comes from the reporting bias as follows:

$$\begin{aligned} (N-1)(F(0)\underline{r} + (1-F(0))\overline{r})^2 &= (N-1)((1-F(0)) - F(0)k(0))^2\overline{r}^2 \\ &= (N-1)\left(\frac{1/2 - F(0)}{(N-1)F(0) + 1/2}\right)^2 \overline{r}^2. \end{aligned}$$

The first equality follows from using that $\underline{r} = -k(0)\overline{r}$. The second equality follows from using that $k(0) = ((N-1)(1-F(0)) + 1/2)/((N-1)F(0) + 1/2)$ and from simplifying. Because $\overline{r}$ does not depend on N, it follows immediately that the expected welfare cost of the reporting bias converges to zero as $N \to \infty$.

Consider now the expected welfare loss which comes from the difference between the socially optimal and the actual reporting behavior:

$$\begin{aligned} & F(0)\mathbf{E}_{X_i}[-(\underline{r} - X_i)^2|X_i \leq 0] + (1 - F(0))\mathbf{E}_{X_i}[-(\overline{r} - X_i)^2|X_i \geq 0] \\ = \; & -\sigma^2 + F(0)(2\underline{r}\mathbf{E}_{X_i}[X_i|X_i \leq 0] - \underline{r}^2) + (1 - F(0))(2\overline{r}\mathbf{E}_{X_i}[X_i|X_i \geq 0] - \overline{r}^2) \\ = \; & -\sigma^2 - F(0)(-2k(0)\mathbf{E}_{X_i}[X_i|X_i \leq 0]\mathbf{E}_{X_i}[X_i|X_i \geq 0] - k(0)^2\mathbf{E}_{X_i}[X_i|X_i \geq 0]^2) \\ & +(1 - F(0))\mathbf{E}_{X_i}[X_i|X_i \geq 0]^2 \\ = \; & -\sigma^2 + F(0) \cdot k(0)F(0)/(1 - F(0)) \cdot (2 - k(0)F(0)/(1 - F(0))) \cdot \mathbf{E}_{X_i}[X_i|X_i \leq 0]^2 \\ & +(1 - F(0))\mathbf{E}_{X_i}[X_i|X_i \geq 0]^2. \end{aligned}$$

The transformations arise as follows: The first equality follows from multiplying out and from using that $\mathbf{E}_{X_i}[X_i^2] = \sigma^2$. The second equality follows from using that $\overline{r} = \mathbf{E}_{X_i}[X_i|X_i \geq 0]$ and that $\underline{r} = -k(0)\mathbf{E}_{X_i}[X_i|X_i \geq 0]$. The third equality follows from using in the second term that $\mathbf{E}_{X_i}[X_i|X_i \geq 0] = -F(0)/(1 - F(0)) \cdot \mathbf{E}_{X_i}[X_i|X_i \leq 0]$ by the normalization of the distribution. The so obtained expression depends only through $k(0)$ on N. Because $k(0) \to (1 - F(0))/F(0)$ as $N \to \infty$, the welfare loss form the difference between the socially optimal and the actual reporting behavior converges to $-\sigma^2 + F(0)\mathbf{E}_{X_i}[X_i|X_i \leq 0]^2 + (1 - F(0))\mathbf{E}_{X_i}[X_i|X_i \geq 0]^2 = -\sigma^2 + c(0)$.

I obtain from the arguments in the preceding two paragraphs

$$\lim_{N\to\infty} V(\phi_a, R^{a,B}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0])) = -\sigma^2 + c(0).$$

Step 2: Asymptotic welfare attained by the optimal binary average mechanism. For any N, let (ϕ_a, R_N) with $R_N := \{\underline{r}_N, \overline{r}_N\}$ be an optimal binary average mechanism. If the induced reporting behavior is non–constant for a given N, there exists by Corollary 2 a threshold t_N such that each agent i reports $\underline{r}_N$ if $x_i < t_N$ and $\overline{r}_N$ if $x_i > t_N$. If the induced reporting behavior is constant, the same is true for either $t_N = \underline{t}$ or $t_N = \overline{t}$. The reporting bias associated to the mechanism (ϕ_a, R_N) is $b_N := F(t_N)\underline{r}_N + (1 - F(t_N))\overline{r}_N$. I first derive four properties which must hold for any sequence of optimal binary average mechanisms. I use these properties then to derive an upper bound on asymptotic welfare and show that the mechanism from Step 1 attains this upper bound asymptotically.

Property 1: $b_N \to 0$ as $N \to \infty$. (B.8) implies $V(\phi_a, R_N) \leq -(N-1)b_N^2$. By optimality of R_N, $V(\phi_a, R_N) \geq V(\phi_a, \{0\}) = -\sigma^2$. It follows $|b_N| \leq \sigma/\sqrt{N-1}$ such that $b_N \to 0$ as $N \to \infty$.

Property 2: The sequence t_N is bounded away from $\underline{t}$ and from $\overline{t}$. The information reflected by the agents' reporting behavior is $(\text{sgn}(X_1 - t_N), \ldots, \text{sgn}(X_N - t_N))$. This together with Lemma 2 (b) implies that $V^{*|(\text{sgn}(X_1-t_N),\ldots,\text{sgn}(X_N-t_N))} = -\sigma^2 + c(t_N)$ constitutes an upper bound on $V(\phi_a, R_N)$. By optimality of R_N, welfare attained by the specific average mechanism $(\phi_a, R^{a,B}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0]))$ constitutes a lower bound on $V(\phi_a, R_N)$. It follows that $-\sigma^2 + c(t_N) \geq V(\phi_a, R^{a,B}(0, \mathbf{E}_{X_i}[X_i|X_i \geq 0]))$ for any N. By Step 1 of this proof, the right–hand side of this inequality converges to $-\sigma^2 + c(0)$ as $N \to \infty$. Since $c(t)$ is continuous with $c(0) > 0$ and $c(\underline{t}) = c(\overline{t}) = 0$, necessary for the inequality to hold is that t_N is bounded away from $\underline{t}$ and $\overline{t}$.

Property 3: The sequences $\underline{r}_N$ and $\overline{r}_N$ are bounded. I present the argument for $\overline{r}_N$, the argument for $\underline{r}_N$ is analogous. Assume to the contrary that $\overline{r}_N \to \infty$ for some subsequence. (B.8) implies that $V(\phi_a, R_N) \leq -(1 - F(t_N))\mathbf{E}_{X_i}[(\overline{r}_N - X_i)^2|X_i \geq t(N)]$. By optimality of R_N, $V(\phi_a, R_N) \geq V(\phi_a, \{0\}) = -\sigma^2$. By putting both inequalities together, $\sigma^2 \geq (1 - F(t_N))\mathbf{E}_{X_i}[(\overline{r}_N - X_i)^2|X_i \geq t_N]$. Since the expected value expression approaches ∞ for the considered subsequence, necessary for the inequality to hold is that $t_N \to \overline{t}$ for the considered subsequence. Since t_N is by Property 2 however bounded away from $\overline{t}$, I obtain a contradiction. Hence, $\overline{r}_N$ must be bounded.

Property 4: $t_N \to 0$ as $N \to \infty$. By Property 2, the optimal binary average mechanism induces a non–constant reporting behavior for any sufficiently large N. This requires that an agent with private signal t_N is indifferent between the report $\underline{r}_N$ and the report $\overline{r}_N$. The indifference condition is by Proposition 1 (b) given by $N(1-\alpha)t_N - (N-1)b_N = (\underline{r}_N + \overline{r}_N)/2$. It can be rewritten as $t_N = (\underline{r}_N + \overline{r}_N)/(2N(1-\alpha)) + (N-1)/N \cdot b_N/(1-\alpha)$. As the right–hand side converges by Properties 1 and 3 to zero as $N \to \infty$, I obtain Property 4.

The information reflected by the agents' reporting behavior is $(\text{sgn}(X_1 - t_N), \ldots, \text{sgn}(X_N - t_N))$. This together with Lemma 2 (b) implies that $V^{*|(\text{sgn}(X_1-t_N),\ldots,\text{sgn}(X_N-t_N))} = -\sigma^2 + c(t_N)$ constitutes an upper bound on $V(\phi_a, R_N)$. Since $c(t_N)$ is continuous, Property 4 implies that $\lim_{N\to\infty} V^{*|(\text{sgn}(X_1-t_N),\ldots,\text{sgn}(X_N-t_N))} = -\sigma^2 + c(0)$. Since my specific average mechanism from Step 1 attains this upper bound asymptotically, I obtain the result. q.e.d.

Proof of Lemma 5.

Lemma 5 follows as a corollary from Lemma 7 (b) below. q.e.d.

Proof of Proposition 4.

Proposition 4 follows directly from Lemma 4, Lemma 5 and Lemma 2 (b) and (c). q.e.d.

Proof of Lemma 6.

Step 1: Welfare attained by the unrestricted average mechanism. I have

$$V(\phi_a, \mathbb{R}) = \mathbf{E}_{X_i}[-(N(1-\alpha)X_i - X_i)^2] = -(N(1-\alpha)-1)^2\sigma^2.$$

The first equality follows from using that the unrestricted average mechanism implies the unbiased reporting behavior $\widehat{r}_i^{a,\mathbb{R}}(x_i) = N(1-\alpha)x_i$ in (6). The second equality follows from using that $\mathbf{E}_{X_i}[X_i^2] = \sigma^2$.

Step 2: A lower bound on welfare attained by the unrestricted median mechanism. I have

$$\begin{aligned} V(\phi_m, \mathbb{R}) &= V^{*|\mathrm{m}(X)} - N \cdot \mathbf{E}_X[(\widehat{r}_i^{m,\mathbb{R}}(\mathrm{m}(X)) - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2] \\ &= V^{*|\mathrm{m}(X)} - N \cdot ((N-1)/N - \alpha)^2 \cdot \mathbf{E}_X[(\mathrm{m}(X) - h(\mathrm{m}(X)))^2]. \end{aligned}$$

The first equality follows from (4). The second equality follows from using the definition of $\widehat{r}_i^{m,\mathbb{R}}$ and of $\widehat{r}_i^{m,*}$. Since $V^{*|\mathrm{m}(X)}$ is not affected by α, $V(\phi_m, \mathbb{R})$ is non–decreasing in α. Hence,

$$\begin{aligned} V(\phi_m, \mathbb{R}) \geq V(\phi_m, \mathbb{R})|_{\alpha=0} &= \mathbf{E}_X[-N(\mathrm{m}(X) - \overline{X})^2] \\ &= -\sigma^2 + 2N\mathbf{E}_X[\mathrm{m}(X)\overline{X}] - N\mathbf{E}_X[\mathrm{m}(X)^2]. \end{aligned}$$

The two equalities arise as follows: The decision function which is implemented when $\alpha = 0$ is $\widehat{y}(x) = \mathrm{m}(x)$. The first equality follows from using this decision function in the original definition of the welfare functional. The second equality follows from multiplying out and from using that $\mathbf{E}_X[N\overline{X}^2] = -\sigma^2$.

I next derive a lower bound on $V(\phi_m, \mathbb{R})|_{\alpha=0}$. By Theorem 2.1 in Bickel (1967), any two order statistics are positively correlated. This implies $\mathbf{E}_X[\mathrm{m}(X)\overline{X}] \geq 0$. By Theorem 1 in Yang (1982), the variance of the median order statistic is smaller than the population variance when the sample size is odd. This implies $\mathbf{E}_X[\mathrm{m}(X)^2] \leq \sigma^2 + \mathbf{E}_X[\mathrm{m}(X)]^2$. Both properties together imply

$$V(\phi_m, \mathbb{R})|_{\alpha=0} \geq -(N+1)\sigma^2 - \mathbf{E}_X[\sqrt{N}\mathrm{m}(X)]^2.$$

Step 3: Comparison. By Step 1 and Step 2, a sufficient condition for $V(\phi_a, \mathbb{R}) < V(\phi_m, \mathbb{R})$ is $(N(1-\alpha)-1)^2 - (N+1) > \mathbf{E}_X[\sqrt{N}\mathrm{m}(X)]^2/\sigma^2$. For any given $\alpha \in [0, 1)$, the left–hand side converges to ∞ as $N \to \infty$. Since the asymptotic distribution of $\sqrt{N}\mathrm{m}(X)$ is normal with mean η (see, e.g., DasGupta (2008)), the right–hand side converges towards η^2/σ^2 as $N \to \infty$. It follows that $V(\phi_m, \mathbb{R}) > V(\phi_a, \mathbb{R})$ whenever N is sufficiently large. This is the first part of the lemma. Furthermore, $V^{*|0} = -\sigma^2$ and Step 1 imply that $V(\phi_a, \mathbb{R}) < V^{*|0}$ whenever $N(1-\alpha) > 2$. This is the second part of the lemma. q.e.d.

Proof of Lemma 7.

(a) Fix any $\alpha \in [0, 1)$ and consider $N \in \mathcal{N}' := \{N \in \mathcal{N} | N > 1/(1-\alpha)\}$. Moreover, consider $\phi = \phi_a$. For any $N \in \mathcal{N}'$, let R_N be an optimal report space from $\mathcal{R}$ and let $\widehat{r}_i^N$ be a sBNE of (ϕ_a, R_N). Suppose without loss of generality that $R_N = \mathrm{cl}(\widehat{r}_i^N(\mathcal{X}))$. Define $\underline{r}_N := \widehat{r}_i^N(\underline{t})$,

$\overline{r}_N := \widehat{r}_i^N(\overline{t})$ and $b_N := \mathbf{E}_{X_i}[\widehat{r}_i^N(X_i)]$. If I knew that the sequences $\underline{r}_N$ and $\overline{r}_N$ must be bounded, the proof would be relatively straightforward. (It would then consist only of Step 3a below.) It is however not trivial that $\underline{r}_N$ cannot become unboundedly small or that $\overline{r}_N$ cannot become unboundedly large as $N \to \infty$. This makes the proof more involved.

My strategy of proof is as follows: I will construct a sequence of report spaces $(R'_N)_{N \in \mathcal{N}'}$ with $R'_N \in \mathcal{R}_B$ such that

$$\forall \epsilon > 0 \; \exists N' \in \mathbb{N} \; \forall N \in \mathcal{N}' \text{ with } N > N' : V(\phi_a, R_N) - V(\phi_a, R'_N) < \epsilon. \tag{B.9}$$

Since it follows from Lemma 4 and Lemma 2 (b) that the welfare assumed by the optimal binary average mechanism converges as $N \to \infty$, it will follow from (B.9) that $V(\phi_a, R_N)$ converges towards the same value. This will prove Part (a) of the lemma.

Step 1: Definition of a class of auxiliary report spaces. For any $\underline{C} \in [\underline{r}_N, b_N]$ and any $\overline{C} \in [b_N, \overline{r}_N]$ define $R_N(\underline{C}, \overline{C}) := (R_N \cup \{\underline{C}, \overline{C}\}) \cap [\underline{C}, \overline{C}]$. Furthermore, define $\widehat{r}_i^N(x_i; \underline{C}, \overline{C}) := \max\{\min\{\widehat{r}_i^N(x_i), \overline{C}\}, \underline{C}\}$ and $b_N(\underline{C}, \overline{C}) := \mathbf{E}_{X_i}[\widehat{r}_i^N(X_i; \underline{C}, \overline{C})]$. $\widehat{r}_i^N(x_i; \underline{C}, \overline{C})$ constitutes a sBNE of $(\phi_a, R_N(\underline{C}, \overline{C}))$ if $b_N(\underline{C}, \overline{C}) = b_N$. Since $\widehat{r}_i^N(x_i; \underline{C}, \overline{C})$ is continuous and weakly increasing in $\underline{C}$ and $\overline{C}$, $b_N(\underline{C}, \overline{C})$ is continuous and weakly increasing in $\underline{C}$ and $\overline{C}$. Since $b_N(\underline{r}_N, \overline{r}_N) = b_N$ and $b_N(b_N, b_N) = b_N$, there exists a continuous and weakly decreasing function $\overline{C} : [\underline{r}_N, b_N] \to [b_N, \overline{r}_N]$ with $\overline{C}(\underline{r}_N) = \overline{r}_N$ and $\overline{C}(b_N) = b_N$ such that $b_N(\underline{C}, \overline{C}(\underline{C})) = b_N$ for all $\underline{C} \in [\underline{r}_N, b_N]$. That is, $\widehat{r}_i^N(x_i; \underline{C}, \overline{C}(\underline{C}))$ constitutes a sBNE of $(\phi_a, R_N(\underline{C}, \overline{C}(\underline{C})))$ for any $\underline{C} \in [\underline{r}_N, b_N]$.

Step 2: For any $N \in \mathcal{N}'$, $R_N \cap (-\infty, \underline{t}) = \emptyset$ or $R_N \cap (\overline{t}, \infty) = \emptyset$. Assume to the contrary that there exists $N \in \mathcal{N}'$ such that $\underline{r}_N < \underline{t}$ and $\overline{r}_N > \overline{t}$ is true. By Step 1, there would then exist a report space $R_N(\underline{C}', \overline{C}(\underline{C}'))$ with $\underline{C}' = \underline{t}$ and $\overline{C}(\underline{C}') \in [\overline{t}, \overline{r}_N]$ or with $\underline{C}' \in [\underline{r}_N, \underline{t}]$ and $\overline{C}(\underline{C}') = \overline{t}$. The reporting behavior implied by $(\phi_a, R_N(\underline{C}', \overline{C}(\underline{C}')))$ lies everywhere weakly closer and on at least one interval of positive length strictly closer to the socially optimal reporting behavior $\widehat{r}_i^{a,*}$ than the reporting behavior implied by (ϕ_a, R_N). Moreover, both mechanisms imply the same reporting bias. By the representation (B.8) of $V(\phi_a, R)$ which I obtain when I derive (6) in this Appendix, this implies $V(\phi_a, R_N) < V(\phi_a, R_N(\underline{C}', \overline{C}(\underline{C}')))$ contradicting the optimality of R_N.

Step 3: Construction of the sequence $(R'_N)_{N \in \mathcal{N}'}$. Step 2 implies that the two sets $\mathcal{N}'_1 := \{N \in \mathcal{N}' | R_N \cap (-\infty, \underline{r}') \neq \emptyset\}$ and $\mathcal{N}'_2 := \{N \in \mathcal{N}' | R_N \cap (\overline{r}', \infty) \neq \emptyset\}$ with $\underline{r}' := \underline{t} - (\overline{t} - \underline{t})$ and $\overline{r}' := \overline{t} + (\overline{t} - \underline{t})$ are mutually exclusive. The three sets $\mathcal{N}'_1$, $\mathcal{N}'_2$ and $\mathcal{N}'_3 := \mathcal{N}' \backslash (\mathcal{N}'_1 \cup \mathcal{N}'_2)$ constitute thus a partition of $\mathcal{N}'$. In the subsequent Steps 3a, 3b and 3c, I construct for any $\iota \in \{1, 2, 3\}$ and any $N \in \mathcal{N}'_\iota$ a report space $R'_N \in \mathcal{R}_B$ such that

$$\forall \epsilon > 0 \; \exists N'_\iota \in \mathbb{N} \; \forall N \in \mathcal{N}'_\iota \text{ with } N > N'_\iota : V(\phi_a, R_N) - V(\phi_a, R'_N) < \epsilon. \tag{B.10}$$

It follows then immediately that also (B.9) must hold. Since (B.10) holds trivially for any ι for which N_ι is finite, it suffices thus to investigate the case in which it is infinite.

Step 3a: Construction of R'_N for $N \in \mathcal{N}'_3$. Consider $N \in \mathcal{N}'_3$ and suppose without loss of generality that $\mathcal{N}'_3$ is not finite. Define $\underline{t}_N := \underline{r}_N/(N(1-\alpha)) + (N-1)/(N(1-\alpha)) \cdot b_N$ and $\overline{t}_N := \overline{r}_N/(N(1-\alpha)) + (N-1)/(N(1-\alpha)) \cdot b_N$ for any $N \in \mathcal{N}'_3$. $x_i \geq \overline{t}_N$ (resp., $x_i \leq \underline{t}_N$) can then be rewritten as $\widehat{r}_i(x_i, b_N) \geq \overline{r}_N$ (resp., $\widehat{r}_i(x_i, b_N) \leq \underline{r}_N$). That is, under mechanism (ϕ_a, R_N), the preferred report of an agent with attribute $x_i \in [\overline{t}_N, \overline{t}]$ (resp. $x_i \in [\underline{t}, \underline{t}_N]$) is larger than $\overline{r}_N$ (resp., smaller than $\underline{r}_N$). This implies $\widehat{r}_i^N(x_i) = \overline{r}_N$ for any $x_i \in [\overline{t}_N, \overline{t}]$ and $\widehat{r}_i^N(x_i) = \underline{r}_N$ for any $x_i \in [\underline{t}, \underline{t}_N]$. From (B.8), I get the following inequality by omitting the term $-(N-1)b_N^2$

from the right–hand side:

$$\begin{aligned} V(\phi_a, R_N) &\leq \text{Prob}_X[X_i \in [\underline{t}, \underline{t}_N)] \cdot \mathbf{E}_X[-(\underline{r}_N - X_i)^2 | X_i \in [\underline{t}, \underline{t}_N)] \\ &\quad +\text{Prob}_X[X_i \in [\underline{t}_N, \overline{t}_N)] \cdot \mathbf{E}_X[-(\widehat{r}_i^N(X_i) - X_i)^2 | X_i \in [\underline{t}_N, \overline{t}_N)] \\ &\quad +\text{Prob}_X[X_i \in [\overline{t}_N, \overline{t}]] \cdot \mathbf{E}_X[-(\overline{r}_N - X_i)^2 | X_i \in [\overline{t}_N, \overline{t}]] \\ &=: \widetilde{V}_a. \end{aligned} \tag{B.11}$$

For any $\epsilon > 0$ and for any sufficiently large $N \in \mathcal{N}_3'$, the following must then be true:

$$\begin{aligned} \widetilde{V}_a &\leq \text{Prob}_X[X_i < 0]\mathbf{E}_X[-(\underline{r}_N - X_i)^2 | X_i < 0] \\ &\quad +\text{Prob}_X[X_i \geq 0]\mathbf{E}_X[-(\overline{r}_N - X_i)^2 | X_i \geq 0] + \epsilon/2 \\ &\leq \max_{r', r'' \in \mathbb{R}} \quad \text{Prob}_X[X_i < 0]\mathbf{E}_X[-(r' - X_i)^2 | X_i < 0] \\ &\qquad\qquad\quad +\text{Prob}_X[X_i \geq 0]\mathbf{E}_X[-(r'' - X_i)^2 | X_i \geq 0] + \epsilon/2 \\ &= V^{*|(\text{sgn}(X_1), \ldots, \text{sgn}(X_N))} + \epsilon/2 \\ &\leq V(\phi_a, R^{a,B}(0, \mathbf{E}_{X_i}[X_i | X_i \geq 0])) + \epsilon. \end{aligned} \tag{B.12}$$

The transformations arise as follows: Note first that $\underline{t}_N, \overline{t}_N \to 0$ (this holds because $\underline{r}_N$ and $\overline{r}_N$ are bounded in the considered case and because optimality of R_N implies $b_N \to 0$ by the reasoning in Property 1 in Step 2 of the Proof to Lemma 4). The first inequality follows since the probability expressions and the expected value expressions in (B.11) are continuous in $\underline{t}_N$ and $\overline{t}_N$. This implies that I do not lose more than $\epsilon/2$ by replacing $\underline{t}_N$ and $\overline{t}_N$ in $\widetilde{V}_a$ by 0 when N is sufficiently large. The second inequality is straightforward. The equality follows from realizing that the maximization problem gives me welfare from optimal centralized decision–making conditional on the information $(\text{sgn}(X_1), \ldots, \text{sgn}(X_N))$. The last inequality follows because Lemma 4 implies that for any $\epsilon > 0$ and any sufficiently large N I have $V^{*|(\text{sgn}(X_1), \ldots, \text{sgn}(X_N))} < V(\phi_a, R^{a,B}(0, \mathbf{E}_{X_i}[X_i | X_i \geq 0])) + \epsilon/2$.

Since (B.11) and (B.12) hold for any ϵ when N is sufficiently large, I can conclude that (B.10) holds for $\iota = 3$ when I set $R_N' := R^{a,B}(0, \mathbf{E}_{X_i}[X_i | X_i \geq 0])$.

Step 3b: Construction of R_N' for $N \in \mathcal{N}_1'$. Consider $N \in \mathcal{N}_1'$ and suppose without loss of generality that $\mathcal{N}_1'$ is not finite. Define $\underline{r}_N' := \min\{r_i \in \text{cl}(\widehat{r}_i^N(\mathcal{X})) | r_i \geq \underline{r}'\}$, $\underline{t}_N' := \underline{r}_N'/(N(1-\alpha)) + (N-1)/(N(1-\alpha)) \cdot b_N$ and $\overline{t}_N := \overline{r}_N/(N(1-\alpha)) + (N-1)/(N(1-\alpha)) \cdot b_N$. These definitions imply three properties.

Property 1: $\mathbf{E}_X[-(\widehat{r}_i^N(X_i) - X_i)^2 | X_i \in [\overline{t}_N, \overline{t}]] = \mathbf{E}_X[-(\overline{r}_N - X_i)^2 | X_i \in [\overline{t}_N, \overline{t}]]$. $x_i \geq \overline{t}_N$ can be rewritten as $\widehat{r}_i(x_i, b_N) \geq \overline{r}_N$. That is, the preferred report of an agent with attribute $x_i \in [\overline{t}_N, \overline{t}]$ under mechanism (ϕ_a, R_N) is larger than $\overline{r}_N$. Since $\overline{r}_N$ is admissible and since there exists no larger admissible report, $\widehat{r}_i^N(x_i) = \overline{r}_N$ for any $x_i \in [\overline{t}_N, \overline{t}]$. This implies Property 1.

Property 2: $\mathbf{E}_X[-(\widehat{r}_i^N(X_i) - X_i)^2 | X_i \in [\underline{t}, \underline{t}_N')] \leq \mathbf{E}_X[-(\underline{r}_N' - X_i)^2 | X_i \in [\underline{t}, \underline{t}_N')]$. $x_i < \underline{t}_N'$ can be rewritten as $\widehat{r}_i(x_i, b_N) < \underline{r}_N'$. That is, the preferred report of an agent with attribute $x_i \in [\underline{t}, \underline{t}_N')$ under mechanism (ϕ_a, R_N) is strictly smaller than $\underline{r}_N'$. Since $\underline{r}_N'$ is by construction admissible, $\widehat{r}_i^N(x_i) \leq \underline{r}_N'$ must be true. By the construction of the set $\mathcal{N}_1'$ and Step 2, $\underline{r}_N' \leq \overline{r}_N \leq \overline{t}$. I have to distinguish two cases: Suppose first $\underline{r}_N' \leq \underline{t}$ and consider $x_i \in [\underline{t}, \underline{t}_N')$. $\underline{r}_N'$ lies then weakly closer to x_i than $\widehat{r}_i^N(x_i)$. This implies Property 2. Suppose next $\underline{r}_N' > \underline{t}$ and consider $x_i \in [\underline{t}, \underline{t}_N')$. $R_N \cap [\underline{r}', \underline{r}_N')$ is then empty by construction of $\underline{r}_N'$. This implies $\widehat{r}_i^N(x_i) \in (-\infty, \underline{r}') \cup \{\underline{r}_N'\}$. Since any report in $\mathcal{X}$ lies by construction of $\underline{r}'$ closer to x_i than any report in $(-\infty, \underline{r}')$, I obtain in particular that $\underline{r}_N'$ lies weakly closer to x_i than $\widehat{r}_i^N(x_i)$. This implies again Property 2.

Property 3: $\underline{t}_N' \to 0$ and $\overline{t}_N \to 0$ as $N \to \infty$. First, note that $\overline{r}_N$ is bounded by construction of $\mathcal{N}_1'$ and Step 2. Second, note that $\underline{r}_N'$ is bounded by construction. Third, note that optimality of R_N implies that $b_N \to 0$ as $N \to \infty$ (see Property 1 in Step 2 of the Proof to Lemma 4). By using this in the definition of $\underline{t}'_N$ and of $\overline{t}_N$, I obtain Property 3.

I get the following inequality from (B.8) by omitting the term $-(N-1)b_N^2$ on the right–hand side and by using Properties 1 and 2:

$$\begin{aligned}
V(\phi_a, R_N) \quad \leq \quad & \mathrm{Prob}_X[X_i \in [\underline{t}, \underline{t}'_N)] \cdot \mathbf{E}_X[-(\underline{r}'_N - X_i)^2 | X_i \in [\underline{t}, \underline{t}'_N)] \\
& + \mathrm{Prob}_X[X_i \in [\underline{t}'_N, \overline{t}_N)] \cdot \mathbf{E}_X[-(\widehat{r}_i^N(X_i) - X_i)^2 | X_i \in [\underline{t}'_N, \overline{t}_N)] \\
& + \mathrm{Prob}_X[X_i \in [\overline{t}_N, \overline{t}]] \cdot \mathbf{E}_X[-(\overline{r}_N - X_i)^2 | X_i \in [\overline{t}_N, \overline{t}]] \\
=: \quad & \widetilde{V}_b
\end{aligned}$$

Since $\widetilde{V}_b$ has the same structure as $\widetilde{V}_a$ (in particular, $\underline{r}'_N$ is bounded), the remainder of Step 3b is analogous to how I proceed with $\widetilde{V}_a$ in Step 3a.

Step 3c: Construction of R'_N for $N \in \mathcal{N}'_2$. This step is analogous to Step 3b.

(b) Suppose without loss of generality $\eta > 0$. Fix any $\alpha \in [0, 1)$ and consider $N \in \mathcal{N}'' := \{N \in \mathcal{N} | N > 8/(1-\alpha)\}$. Moreover, consider $\phi = \phi_m$. For any $N \in \mathcal{N}''$, let R_N be an optimal report space from $\mathcal{R}$ and let $\widehat{r}_i^N$ be a sBNE of (ϕ_a, R_N). Suppose without loss of generality that $R_N = \mathrm{cl}(\widehat{r}_i^N(\mathcal{X}))$. Define $R^m := (7/8 \cdot (1-\alpha)\eta, 9/8 \cdot (1-\alpha)\eta)$ and $R^{m,*} := (-1/8 \cdot (1-\alpha)\eta, 1/8 \cdot (1-\alpha)\eta)$. Note that $\widehat{r}_i^{m,*}(\eta) \in R^{m,*}$ and $\widehat{r}_i^m(\eta) \in R^m$ for any $N \in \mathcal{N}''$. Since $\widehat{r}_i^{m,*}$ and $\widehat{r}_i^m$ are continuous, there exists $\mathcal{X}_\epsilon := (\eta - \epsilon, \eta + \epsilon)$ with $\epsilon > 0$ such that $\widehat{r}_i^{m,*}(\mathcal{X}_\epsilon) \subset R^{m,*}$ and $\widehat{r}_i^m(\mathcal{X}_\epsilon) \in R^m$ for any $N \in \mathcal{N}''$. Consider in the remainder of this proof any fix ϵ for which this the case.

Property 1: For any sufficiently large N, $R_N \cap R^m = \emptyset$. By omitting the "additional loss" which occurs when $\mathrm{m}(X) \notin \mathcal{X}_\epsilon$ on the right–hand side of (4), I obtain

$$\begin{aligned}
V(\phi_m, R_N) \quad \leq \quad & V^{*|\mathrm{m}(X)} + N \cdot \mathrm{Prob}_X[\mathrm{m}(X) \in \mathcal{X}_\epsilon] \\
& \cdot \mathbf{E}_X[-(\widehat{r}_i^N(\mathrm{m}(X)) - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2 | \mathrm{m}(X) \in \mathcal{X}_\epsilon].
\end{aligned}$$

Assume to the contrary that $R_N \cap R^m \neq \emptyset$. It follows then from Proposition 2 (c) that $\widehat{r}_i^N(\mathrm{m}(X)) \in (5/8 \cdot (1-\alpha)\eta, 11/8 \cdot (1-\alpha)\eta)$ whenever $\widehat{r}_i^m(\mathrm{m}(X)) \in R^m$. This and the construction of R^m, $R^{m,*}$ and $\mathcal{X}_\epsilon$ implies that the distance between the socially optimal decision and the selected decision is at least $4/8 \cdot (1-\alpha)\eta$ when $\mathrm{m}(X) \in \mathcal{X}_\epsilon$. Moreover, since the distribution of $\mathrm{m}(X)$ gets more and more concentrated around η as $N \to \infty$, I have for any $\epsilon' > 0$ that $\mathrm{Prob}_X[\mathrm{m}(X) \in \mathcal{X}_\epsilon] > 1 - \epsilon'$ when N is sufficiently large. Hence, for any fix $\epsilon' > 0$ I have

$$V(\phi_m, R_N) \quad \leq \quad V^{*|\mathrm{m}(X)} - N \cdot (1 - \epsilon') \cdot (1/2 \cdot (1-\alpha)\eta)^2$$

when N is sufficiently large. Since the right–hand side converges towards $-\infty$ as $N \to \infty$, $V(\phi_m, R_N) < -\sigma^2 = V(\phi_m, \{0\})$ for any sufficiently large N. Since this contradicts the optimality of R_N, I obtain for any sufficiently large N a contradiction to $R_N \cap R^m \neq \emptyset$.

Property 2: For any sufficiently large N, $\mathbf{E}_X[-(\widehat{r}_i^N(\mathrm{m}(X)) - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2 | \mathrm{m}(X) \in \mathcal{X}_\epsilon] \leq \max_{r'_N \in \mathbb{R}} \mathbf{E}_X[-(r'_N - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2 | \mathrm{m}(X) \in \mathcal{X}_\epsilon]$. By Property 1, $R_N \cap R^m = \emptyset$ when N is sufficiently large. For any such N, $\widehat{r}_i^N(\mathrm{m}(x))$ selects for $\mathrm{m}(X) \in \mathcal{X}_\epsilon$ either the largest admissible report below $7/8 \cdot (1-\alpha)\eta$ or the smallest admissible report above $9/8 \cdot (1-\alpha)\eta$. If the same report is chosen whenever $\mathrm{m}(X) \in \mathcal{X}_\epsilon$, Property 2 holds trivially. Suppose thus that not always the same report is chosen. There exists then $x'_i \in \mathcal{X}_\epsilon$ such that $\widehat{r}_i^m(x_i) = r'$ if $x_i < x'_i$ and $\widehat{r}_i^m(x_i) = r'' > r'$ if $x_i > x'_i$. Necessary for this is $\widehat{r}_i^m(x'_i) = (r' + r'')/2$. Since $\widehat{r}_i^{m,*}(x_i) < \widehat{r}_i^m(x'_i)$ for any $x_i, x'_i \in \mathcal{X}_\epsilon$ by construction of $\mathcal{X}_\epsilon$, $\widehat{r}_i^{m,*}(x_i) < (r' + r'')/2$ for any $x_i \in \mathcal{X}_\epsilon$. That is, the planner strictly prefers r' over r'' whenever $\mathrm{m}(X) \in \mathcal{X}_\epsilon$. This implies again Property 2.

Property 3: $\lim_{N\to\infty} N \cdot Prob_X[\mathrm{m}(X) \notin \mathcal{X}_\epsilon] = 0$. Define $S_i := \mathbf{I}_{X_i \geq \eta+\epsilon}$, $S := (S_1, \ldots, S_N)$ and $\overline{S} := 1/N \cdot \sum_{i=1}^N S_i$. Moreover, note that $\mathbf{E}_S[\overline{S}] = 1 - F(\eta+\epsilon)$. I obtain then

$$\begin{aligned}
\mathrm{Prob}_X[\mathrm{m}(X) \geq \eta+\epsilon] &= \mathrm{Prob}_S[\overline{S} \geq (N+1)/(2N)] \\
&= \mathrm{Prob}_S[\overline{S} - \mathbf{E}_S[\overline{S}] \geq F(\eta+\epsilon) - 1/2 + 1/(2N)] \\
&\leq \exp\left(-\frac{2N^2(F(\eta+\epsilon) - 1/2 + 1/(2N))^2}{\sum_{i=1}^N (1-0)^2}\right) \\
&\leq \exp\left(-2N(F(\eta+\epsilon) - 1/2)^2\right)
\end{aligned}$$

The transformations arise as follows: The first two equalities follow straightforwardly from the definition of S_i and of $\overline{S}$. The first inequality follows from applying Hoffeding's Inequality. The second inequality follows from simplifying and using that the value of the exponential function increases when I replace $(F(\eta+\epsilon) - 1/2 + 1/(2N))^2$ by the smaller value $(F(\eta+\epsilon) - 1/2)^2$. Since the right–hand side converges exponentially fast towards 0 as $N \to \infty$ also $\mathrm{Prob}_X[\mathrm{m}(X) \geq \eta+\epsilon]$ must converge exponentially fast towards 0. This implies that $N \cdot \mathrm{Prob}_X[\mathrm{m}(X) \geq \eta+\epsilon] \to 0$ as $N \to \infty$. Since an analogous reasoning applies for $\mathrm{Prob}_X[\mathrm{m}(X) \leq \eta - \epsilon]$, I obtain Property 3.

Property 4: $\lim_{N\to\infty} \sqrt{N}\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X) \in \mathcal{X}_\epsilon] = 0$. I have

$$\begin{aligned}
&\lim_{N\to\infty} \sqrt{N}\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X) \in \mathcal{X}_\epsilon] \\
= \; &\lim_{N\to\infty} \sqrt{N}\frac{\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))] - \mathrm{Prob}_X[\mathrm{m}(X) \notin \mathcal{X}_\epsilon]\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X) \notin \mathcal{X}_\epsilon]}{\mathrm{Prob}_X[\mathrm{m}(X) \in \mathcal{X}_\epsilon]} \\
= \; &\lim_{N\to\infty} \sqrt{N}\mathrm{Prob}_X[\mathrm{m}(X) \notin \mathcal{X}_\epsilon]\frac{-\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X) \notin \mathcal{X}_\epsilon]}{\mathrm{Prob}_X[\mathrm{m}(X) \in \mathcal{X}_\epsilon]}.
\end{aligned}$$

The transformations arise as follows: The first equality follows from using that $\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))] = \mathrm{Prob}_X[\mathrm{m}(X) \in \mathcal{X}_\epsilon]\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X) \in \mathcal{X}_\epsilon] + \mathrm{Prob}_X[\mathrm{m}(X) \notin \mathcal{X}_\epsilon]\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X) \notin \mathcal{X}_\epsilon]$. The second equality follows from using that $\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))] = \mathbf{E}_X[\overline{X}] = 0$ by (3), the Law of Iterated Expectations and the normalization of the distributions. Property 4 follows then from noting that Property 3 implies that $\sqrt{N}\mathrm{Prob}_X[\mathrm{m}(X) \notin \mathcal{X}_\epsilon] \to 0$ as $N \to \infty$ and that the fraction expression is bounded.

I can use the derived properties to obtain for any sufficiently large N the following:

$$\begin{aligned}
V(\phi_m, R_N) &= N\mathbf{E}_X[-(\widehat{r}_i^N(\mathrm{m}(X)) - \overline{X})^2] \\
&= -N\mathbf{E}_X[\overline{X}^2] + N\mathbf{E}_X[2\widehat{r}_i^N(\mathrm{m}(X))\mathbf{E}_X[\overline{X}|\mathrm{m}(X)]] - N\mathbf{E}_X[\widehat{r}_i^N(\mathrm{m}(X))^2] \\
&= V^{*|0} + N\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))^2] + N\mathbf{E}_X[-(\widehat{r}_i^N(\mathrm{m}(X)) - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2] \\
&\leq V^{*|0} + N\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))^2] \\
&\quad + N\mathrm{Prob}_X[\mathrm{m}(X) \in \mathcal{X}_\epsilon]\mathbf{E}_X[-(\widehat{r}_i^N(\mathrm{m}(X)) - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2|\mathrm{m}(X) \in \mathcal{X}_\epsilon] \\
&\leq V^{*|0} + N\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))^2] \\
&\quad + N\mathrm{Prob}_X[\mathrm{m}(X) \in \mathcal{X}_\epsilon]\max_{r'_N \in \mathbb{R}} \mathbf{E}_X[-(r'_N - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2|\mathrm{m}(X) \in \mathcal{X}_\epsilon] \\
&= V^{*|0} + N\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))^2] + N\mathrm{Prob}_X[\mathrm{m}(X) \in \mathcal{X}_\epsilon] \\
&\quad \cdot\mathbf{E}_X[-(\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X) \in \mathcal{X}_\epsilon] - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2|\mathrm{m}(X) \in \mathcal{X}_\epsilon] \\
&= V^{*|0} + N\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))^2]
\end{aligned}$$

$$
\begin{aligned}
& -N\mathrm{Prob}_X[\mathrm{m}(X)\in\mathcal{X}_\epsilon]\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))^2|\mathrm{m}(X)\in\mathcal{X}_\epsilon] \\
& +N\mathrm{Prob}_X[\mathrm{m}(X)\in\mathcal{X}_\epsilon]\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X)\in\mathcal{X}_\epsilon]^2 \\
\leq\; & V^{*|0}+N\mathrm{Prob}_X[\mathrm{m}(X)\notin\mathcal{X}_\epsilon]\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))^2|\mathrm{m}(X)\notin\mathcal{X}_\epsilon] \\
& +N\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X)\in\mathcal{X}_\epsilon]^2
\end{aligned}
$$

The transformations arise as follows: The first equality is the original definition of the welfare functional. The second equality follows from multiplying out and applying the Law of Iterated Expectations to the middle term. The third equality follows from using that $-N\mathbf{E}_X[\overline{X}^2]=V^{*|0}$, from adding and subtracting $N\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))^2]$, from using that $\mathbf{E}_X[\overline{X}|\mathrm{m}(X)]=\widehat{r}_i^{m,*}(\mathrm{m}(X))$, and from simplifying. The first inequality follows from omitting the loss which arises from the last term when $\mathrm{m}(X)\notin\mathcal{X}_\epsilon$. The second inequality holds for any sufficiently large N and follows from Property 2. The fourth equality follows since the last expected value expression is maximized by $r'_N=\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X)\in\mathcal{X}_\epsilon]$ by a reasoning like in Lemma 1. The fifth equality follows from multiplying out and simplifying. The third inequality follows from replacing the probability in the last term by 1 and by consolidating the preceding two terms.

Since $\mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))^2|\mathrm{m}(X)\notin\mathcal{X}_\epsilon]$ is bounded, Property 3 implies that the second term on the right–hand side converges to zero as $N\to\infty$. By Property 4, the third term on the right–hand side converges also to zero as $N\to\infty$. $V(\phi_m,R_N)$ is thus smaller than something which converges to $V^{*|0}$. On the hand, optimality of R_N implies that $V(\phi_m,R_N)\geq V^{*|0}$ for any N. Hence, $\lim_{N\to\infty}V(\phi_m,R_N)\to V^{*|0}$ as $N\to\infty$. q.e.d.

Proof of Lemma 8.

By Lemma 2 (b), $V^{*|(\mathrm{sgn}(X_1),\ldots,\mathrm{sgn}(X_N))}=-\sigma^2+c(0)$. Since $X_i\sim\mathrm{U}[-\overline{t},\overline{t}]$ implies $\mathbf{E}_{X_i}[X_i|X_i\geq 0]=\overline{t}/2$ and $\mathbf{E}_{X_i}[X_i|X_i\leq 0]=-\overline{t}/2$, $V^{*|(\mathrm{sgn}(X_1),\ldots,\mathrm{sgn}(X_N))}=-\sigma^2+1/4\cdot\overline{t}^2$. By Lemma 2 (f), $V^{*|\mathrm{m}(X)}=-\sigma^2+1/4\cdot(N+1)^2/((N+1)^2-1)\cdot\overline{t}^2$. Hence, $V^{*|\mathrm{m}(X)}>V^{*|(\mathrm{sgn}(X_1),\ldots,\mathrm{sgn}(X_N))}$.q.e.d.

Proof of Proposition 5.

Step 1: A lower bound on welfare attained by the optimal average mechanism. Consider the mechanism $(\phi_a,R^{a,I}(t',t''))$ with $t''=\overline{t}/(2N(1-\alpha))$ and $t'=-t''$. I have then $b^{a,I}(t',t'')=0$ and $R^{a,I}(t',t'')=[-\overline{t}/2,\overline{t}/2]$. By Corollary 1 (a), the unique sBNE of the game implied by this mechanism is given by $\widehat{r}_i(x_i)=\overline{t}/2$ if $x_i>t''$, $\widehat{r}_i(x_i)=N(1-\alpha)x_i$ if $x_i\in[t',t'']$ and $\widehat{r}_i(x_i)=-\overline{t}/2$ if $x_i<t'$. By (6) and symmetry of distribution and reporting behavior,

$$
\begin{aligned}
V(\phi_a,R^{a,I}(t',t'')) &= -2\int_0^{\overline{t}/(2N(1-\alpha))}(N(1-\alpha)x_i-x_i)^2\cdot 1/(2\overline{t})\cdot\mathrm{d}x_i \\
&\quad -2\int_{\overline{t}/(2N(1-\alpha))}^{\overline{t}}(\overline{t}/2-x_i)^2\cdot 1/(2\overline{t})\cdot\mathrm{d}x_i.
\end{aligned}
$$

Moreover, by applying Leibniz' Rule,

$$
\frac{\mathrm{d}}{\mathrm{d}\alpha}V(\phi_a,R^{a,I}(t',t'')) = -2\int_0^{\overline{t}/(2N(1-\alpha))}2(-Nx_i)\cdot(N(1-\alpha)x_i-x_i)\cdot 1/(2\overline{t})\cdot\mathrm{d}x_i.
$$

Because $N(1-\alpha)\geq 1$, I obtain that $V(\phi_a,R^{a,I}(t',t''))$ is non–decreasing in α. A lower bound on $V(\phi_a, R^{a,I}(t', t''))$ is thus

$$\begin{aligned}
& V(\phi_a, [-\bar{t}/2, \bar{t}/2])\big|_{\alpha=0} \\
= & -2\int_0^{\bar{t}/(2N)} (Nx_i - x_i)^2 \cdot 1/(2\bar{t}) \cdot \mathrm{d}x_i - 2\int_{\bar{t}/(2N)}^{\bar{t}} (\bar{t}/2 - x_i)^2 \cdot 1/(2\bar{t}) \cdot \mathrm{d}x_i \\
= & -\frac{1}{24}\frac{2N^2 - 2N + 1}{N^2} \cdot \bar{t}^2 = -\sigma^2 + \frac{1}{24}\frac{6N^2 + 2N - 1}{N^2} \cdot \bar{t}^2
\end{aligned}$$

The second equality follows from computing the integrals and from simplifying. The third equality follows from using that $\sigma^2 = \bar{t}^2/3$ to rewrite the welfare formula. This proves that $\max_{R\in\mathcal{R}} V(\phi_a, R) \geq -\sigma^2 + 1/24 \cdot (6N^2 + 2N - 1)/N^2 \cdot \bar{t}^2$.

Step 2: An upper bound on welfare attained by the optimal median mechanism. By Proposition 2 (c), any sBNE $\widehat{r}_i$ of the game implied by any median mechanism exhibits a monotonic reporting behavior. This implies that $\mathrm{m}(\widehat{r}_i(x_1), \ldots, \widehat{r}_i(x_N)) = \widehat{r}_i(\mathrm{m}(x))$. $V^{*|\mathrm{m}(X)}$ constitutes thus an upper bound on $V(\phi_m, R)$. Hence, by Lemma 2 (f), $\max_{R\in\mathcal{R}} V(\phi_m, R) \leq -\sigma^2 + 1/4 \cdot (N+1)^2/((N+1)^2 - 1) \cdot \bar{t}^2$.

Step 3: Comparison of the bounds on welfare derived in Steps 1 and 2. By Step 1 and 2, a sufficient condition for $\max_{R\in\mathcal{R}} V(\phi_a, R) > \max_{R\in\mathcal{R}} V(\phi_m, R)$ is $1/24\cdot(6N^2+2N-1)/N^2 > 1/4\cdot(N+1)^2/((N+1)^2-1)$. After simplifying, the sufficient condition becomes $(2N+1)(N-2) > 0$. Since this condition holds for any $N \geq 3$ and any $\alpha \in [0, (N-1)/N]$, I obtain that the optimal average mechanism outperforms the optimal median mechanism.

Step 4: Derivation of the optimal average mechanism. I am now interested in the report space which solves problem (7). I first extend a result from Melumad and Shibano (1991) and explain then how it relates to my design problem. I use the notation of Melumad and Shibano (1991) (indexed by MS) to explain the extension and explain then how the findings translate into my notation.

Step 4.1: Extension of a result by Melumad and Shibano (1991). Melumad and Shibano (1991) solve the delegation problem where the agent takes the decision $x_{\mathrm{MS}} \in \mathbb{R}$ and his information t_{MS} is uniformly distributed on $[0,1]$. Payoffs are normalized to $-(x_{\mathrm{MS}} - t_{\mathrm{MS}})^2$ for the agent and to $-(x_{\mathrm{MS}} - k_{\mathrm{MS}} - a_{\mathrm{MS}}t_{\mathrm{MS}})^2$ with $k_{\mathrm{MS}}, a_{\mathrm{MS}} \in \mathbb{R}$ for the planner. For me the case with $a_{\mathrm{MS}} \in (0,1)$ will be relevant. For this case, their Proposition 2 implies that an interval delegation set $[\underline{t}_{\mathrm{MS}}(k_{\mathrm{MS}}), \bar{t}_{\mathrm{MS}}(k_{\mathrm{MS}})]$ is optimal. Their Proposition 3 implies that the optimal report space is only not a singleton if $k_{\mathrm{MS}} \in (-a_{\mathrm{MS}}/2, 1 - a_{\mathrm{MS}}/2)$. Moreover, $\underline{t}_{\mathrm{MS}}(k_{\mathrm{MS}}) = 0$ if $k_{\mathrm{MS}} \in (-a_{\mathrm{MS}}/2, 0]$ and $\underline{t}_{\mathrm{MS}}(k_{\mathrm{MS}}) = 2k_{\mathrm{MS}}/(2 - a_{\mathrm{MS}})$ if $k_{\mathrm{MS}} \in (0, 1 - a_{\mathrm{MS}}/2)$. Finally, $\bar{t}_{\mathrm{MS}}(k_{\mathrm{MS}}) = (2k_{\mathrm{MS}} + a_{\mathrm{MS}})/(2 - a_{\mathrm{MS}})$ if $k_{\mathrm{MS}} \in (-a_{\mathrm{MS}}/2, 1 - a_{\mathrm{MS}})$ and $\underline{t}_{\mathrm{MS}}(k_{\mathrm{MS}}) = 1$ if $k_{\mathrm{MS}} \in [1 - a_{\mathrm{MS}}, 1 - a_{\mathrm{MS}}/2)$.

The planner's expected payoff from the optimal report space is given by

$$\begin{aligned}
U_{\mathrm{MS}}(k_{\mathrm{MS}}) \; := \; & -\int_0^{\underline{t}_{\mathrm{MS}}(k_{\mathrm{MS}})} (\underline{t}_{\mathrm{MS}}(k_{\mathrm{MS}}) - k_{\mathrm{MS}} - a_{\mathrm{MS}}t_{\mathrm{MS}})^2 \mathrm{d}t_{\mathrm{MS}} \\
& -\int_{\underline{t}_{\mathrm{MS}}(k_{\mathrm{MS}})}^{\bar{t}_{\mathrm{MS}}(k_{\mathrm{MS}})} (t_{\mathrm{MS}} - k_{\mathrm{MS}} - a_{\mathrm{MS}}t_{\mathrm{MS}})^2 \mathrm{d}t_{\mathrm{MS}} \\
& -\int_{\bar{t}_{\mathrm{MS}}(k_{\mathrm{MS}})}^{1} (\bar{t}_{\mathrm{MS}}(k_{\mathrm{MS}}) - k_{\mathrm{MS}} - a_{\mathrm{MS}}t_{\mathrm{MS}})^2 \mathrm{d}t_{\mathrm{MS}}.
\end{aligned}$$

Important for me is the following question: If the planner could choose a parameter k_{MS}, which one would he prefer? Parameters outside $(-a_{\text{MS}}/2, 1-a_{\text{MS}}/2)$ are clearly dominated as they induce uninformed decision–making by the planner. For all other parameters, I obtain by applying Leibnitz' rule:

$$\begin{aligned}
U'_{\text{MS}}(k_{\text{MS}}) &:= -2(\underline{t}'_{\text{MS}}(k_{\text{MS}})-1)\int_0^{\underline{t}_{\text{MS}}(k_{\text{MS}})}(\underline{t}_{\text{MS}}(k_{\text{MS}})-k_{\text{MS}}-a_{\text{MS}}t_{\text{MS}})\mathrm{d}t_{\text{MS}} \\
&\quad +2\int_{\underline{t}_{\text{MS}}(k_{\text{MS}})}^{\overline{t}_{\text{MS}}(k_{\text{MS}})}(t_{\text{MS}}-k_{\text{MS}}-a_{\text{MS}}t_{\text{MS}})\mathrm{d}t_{\text{MS}} \\
&\quad -2(\overline{t}'_{\text{MS}}(k_{\text{MS}})-1)\int_{\overline{t}_{\text{MS}}(k_{\text{MS}})}^{1}(\overline{t}_{\text{MS}}(k_{\text{MS}})-k_{\text{MS}}-a_{\text{MS}}t_{\text{MS}})\mathrm{d}t_{\text{MS}} \\
&= \frac{\underline{t}'_{\text{MS}}(k_{\text{MS}})-1}{a_{\text{MS}}}\left[(\underline{t}_{\text{MS}}(k_{\text{MS}})-k_{\text{MS}}-a_{\text{MS}}t_{\text{MS}})^2\right]_0^{\underline{t}_{\text{MS}}(k_{\text{MS}})} \\
&\quad +\frac{1}{1-a_{\text{MS}}}\left[((1-a_{\text{MS}})t_{\text{MS}}-k_{\text{MS}})^2\right]_{\underline{t}_{\text{MS}}(k_{\text{MS}})}^{\overline{t}_{\text{MS}}(k_{\text{MS}})} \\
&\quad +\frac{\overline{t}'_{\text{MS}}(k_{\text{MS}})-1}{a_{\text{MS}}}\left[(\overline{t}_{\text{MS}}(k_{\text{MS}})-k_{\text{MS}}-a_{\text{MS}}t_{\text{MS}})^2\right]_{\overline{t}_{\text{MS}}(k_{\text{MS}})}^{1}.
\end{aligned}$$

I argue now that only the middle term may differ from zero. Consider the first term and suppose $k_{\text{MS}} \in (-a_{\text{MS}}/2, 0]$. I have then $\underline{t}_{\text{MS}}(k_{\text{MS}})=0$ such that $[(\underline{t}_{\text{MS}}(k_{\text{MS}})-k_{\text{MS}}-a_{\text{MS}}t_{\text{MS}})^2]_0^{\underline{t}_{\text{MS}}(k_{\text{MS}})}$ is clearly zero. Consider thus the other case in which $k_{\text{MS}} \in (0, 1-a_{\text{MS}}/2)$. I have then $[(\underline{t}_{\text{MS}}(k_{\text{MS}})-k_{\text{MS}}-a_{\text{MS}}t_{\text{MS}})^2]_0^{\underline{t}_{\text{MS}}(k_{\text{MS}})} = ((1-a_{\text{MS}})\underline{t}_{\text{MS}}(k_{\text{MS}})-k_{\text{MS}})^2-(\underline{t}_{\text{MS}}(k_{\text{MS}})-k_{\text{MS}})^2$. By using $\underline{t}_{\text{MS}}(k_{\text{MS}}) = 2k_{\text{MS}}/(2-a_{\text{MS}})$ and simplifying, I obtain again that this is zero. Since a similar reasoning implies that $\left[(\overline{t}_{\text{MS}}(k_{\text{MS}})-k_{\text{MS}}-a_{\text{MS}}t_{\text{MS}})^2\right]_{\overline{t}_{\text{MS}}(k_{\text{MS}})}^{1}=0$, the first term and the third term are for any $k_{\text{MS}} \in (-a_{\text{MS}}/2, 1-a_{\text{MS}}/1)$ both zero. I obtain

$$U'_{\text{MS}}(k_{\text{MS}}) = \frac{1}{1-a_{\text{MS}}}\left(((1-a_{\text{MS}})\overline{t}_{\text{MS}}(k_{\text{MS}})-k_{\text{MS}})^2-((1-a_{\text{MS}})\underline{t}_{\text{MS}}(k_{\text{MS}})-k_{\text{MS}})^2\right).$$

I need now to distinguish three cases.

Case 1: $k_{MS} \in (0, 1-a_{MS})$. By using the structure of $\underline{t}_{\text{MS}}(k_{\text{MS}})$ and $\overline{t}_{\text{MS}}(k_{\text{MS}})$ and simplifying, I obtain $U'_{\text{MS}}(k_{\text{MS}}) = a_{\text{MS}}^2/(4-2a_{\text{MS}})\cdot((1-a_{\text{MS}})/2-k_{\text{MS}})$ and $U''_{\text{MS}}(k_{\text{MS}})<0$. Hence, $k^{*1}_{\text{MS}} = (1-a_{\text{MS}})/2$ is optimal within the considered region of parameters.

Case 2: $k_{MS} \in (-a_{MS}/2, 0]$. By using that $\underline{t}_{\text{MS}}(k_{\text{MS}})=0$ for such parameters, I get $U'_{\text{MS}}(k_{\text{MS}}) = 1/(1-a_{\text{MS}})\cdot(((1-a_{\text{MS}})\overline{t}_{\text{MS}}(k_{\text{MS}})-k_{\text{MS}})^2-k_{\text{MS}}^2)$. By multiplying out and simplifying, this becomes $U'_{\text{MS}}(k_{\text{MS}}) = 1/(1-a_{\text{MS}})\cdot((1-a_{\text{MS}})^2\overline{t}_{\text{MS}}(k_{\text{MS}})^2-2(1-a_{\text{MS}})\overline{t}_{\text{MS}}(k_{\text{MS}})k_{\text{MS}})$. Since $k_{\text{MS}} \le 0$ in the considered region and since $k_{\text{MS}} > -a_{\text{MS}}/2$ implies $\overline{t}_{\text{MS}}(k_{\text{MS}})>0$, I have $U'_{\text{MS}}(k_{\text{MS}}) \ge 0$ for any $k_{\text{MS}} \in (-a_{\text{MS}}/2, 0]$. Hence, I get a corner solution at $k^{*2}_{\text{MS}}=0$.

Case 3: $k_{MS} \in [1-a_{MS}, 1-a_{MS}/2)$. By a reasoning which is analogous to that in Case 2, I have $U'_{\text{MS}}(k_{\text{MS}}) \le 0$ for any $k_{\text{MS}} \in [1-a_{\text{MS}}, 1-a_{\text{MS}}/2)$ such that I get a corner solution at $k^{*3}_{\text{MS}} = 1-a_{\text{MS}}$.

It follows directly from these three case and continuity of $U_{\text{MS}}(k_{\text{MS}})$ that the parameter $k^*_{\text{MS}} = (1-a_{\text{MS}})/2$ is optimal for the planner. The delegation set which is optimal for this parameter is

$$[\underline{t}_{\text{MS}}(k^*_{\text{MS}}), \overline{t}_{\text{MS}}(k^*_{\text{MS}})] = [(1-a_{\text{MS}})/(2-a_{\text{MS}}), 1/(2-a_{\text{MS}})]. \tag{B.13}$$

Step 4.2: Application of Step 4.1 to my problem. Consider now the relaxed version of my problem (7) where the constraint $\mathbf{E}_{X_i}[\widehat{r}_i(X_i)] = b$ is ignored. b assumes then the role of a parameter which the planner can choose freely. Moreover, when I define $t_{\mathrm{MS}}(x_i) := (x_i + \overline{t})/(2\overline{t})$ and $x_{\mathrm{MS}}(r_i) := (r_i + N(1-\alpha)\overline{t} + (N-1)b)/(N(1-\alpha)2\overline{t})$, I can write

$$\begin{aligned} & -(r_i - \widehat{r}_i^a(x_i, b))^2 \\ = \quad & -(x_{\mathrm{MS}}(r_i) - t_{\mathrm{MS}}(x_i))^2 \cdot (N(1-\alpha)2\overline{t})^2 \end{aligned}$$

and

$$\begin{aligned} & -(r_i - (\widehat{r}_i^{a,*}(x_i) - (\sqrt{N}-1)b))^2 \\ = \quad & -\left(x_{\mathrm{MS}}(r_i) - \frac{(N(1-\alpha)-1)\overline{t} + (N-\sqrt{N})b}{N(1-\alpha)2\overline{t}} - \frac{1}{N(1-\alpha)} t_{\mathrm{MS}}(x_i)\right)^2 \cdot (N(1-\alpha)2\overline{t})^2. \end{aligned}$$

Hence, my relaxed problem corresponds to the problem in Melumad and Shibano (1991) with

$$k_{\mathrm{MS}} := \frac{(N(1-\alpha)-1)\overline{t} + (N-\sqrt{N})b}{N(1-\alpha)2\overline{t}} \text{ and } a_{\mathrm{MS}} := \frac{1}{N(1-\alpha)}. \tag{B.14}$$

This allows me to directly apply the result in Step 4.1. The value of b which is optimal in the relaxed problem solves $k_{\mathrm{MS}} = (1 - a_{\mathrm{MS}})/2$. By substituting (B.14) into this condition, I obtain $b^* = 0$. Moreover, by (B.13), the report space $[\underline{r}^*, \overline{r}^*]$ which solves the relaxed problem is implicitly defined by

$$[x_{\mathrm{MS}}(\underline{r}^*), x_{\mathrm{MS}}(\overline{r}^*)] = [(1-a_{\mathrm{MS}})/(2-a_{\mathrm{MS}}), 1/(2-a_{\mathrm{MS}})].$$

By using the definition of $x_{\mathrm{MS}}(r_i)$ and $b = 0$, I obtain after simplifying

$$[\underline{r}^*, \overline{r}^*] = \left[-\frac{N(1-\alpha)}{2N(1-\alpha)-1}\overline{t}, \frac{N(1-\alpha)}{2N(1-\alpha)-1}\overline{t}\right]. \qquad \text{q.e.d.}$$

Since this report space is symmetric about zero, it induces a reporting behavior which is symmetric about zero. Since this means that the ignored constraint is satisfied, the report space which solves the relaxed problem solves also the original problem (7). By using the notation introduced in Corollary 1, I can write this report space as $R^{a,I}(t', t'')$ with $t'' = \overline{t}/(2N(1-\alpha)-1)$ and $t' = -t''$.

Proof of Proposition 6.

Part a: The optimal aggregation rule for $F \in \mathcal{F}_s \cap \mathcal{F}_h$. For the considered class of distributions, Lemma 4 and Lemma 2 (d) imply $\lim_{N\to\infty} \max_{R\in\mathbb{R}} V(\phi_a, R) = \lim_{N\to\infty} V^{*|\mathrm{m}(X)}$. I prove Part a of the result by showing that welfare attained by optimal median mechanisms is bounded above away from $V^{*|\mathrm{m}(X)}$ for large N.

Fix any $\alpha \in [0, 1)$ and consider $N \in \mathcal{N}' := \{N \in \mathcal{N} | N > 1/(1-\alpha)\}$. Moreover, consider $\phi = \phi_m$. For any $N \in \mathcal{N}'$, let R_N be an optimal report space from $\mathcal{R}$ and let $\widehat{r}_i^N$ be a sBNE* of (ϕ_m, R_N). By (4), $V(\phi_m, R_N) = V^{*|\mathrm{m}(X)} + N \cdot \mathbf{E}_X[-(\widehat{r}_i^N(\mathrm{m}(X)) - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2]$. To prove the result, it suffices for me to show that $N \cdot \mathbf{E}_X[-(\widehat{r}_i^N(\mathrm{m}(X)) - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2]$ is bounded away from zero for large N. Before I present the main argument, I prove five properties. Define for this $\gamma^* := h'(0)/((1-\alpha) + \alpha h'(0))$ and note that $\gamma^* \in (0, 1)$ by Lemma A1 (b) in Appendix A.

Property 1: If $\gamma \in (\gamma^*, 1)$, then $\widehat{r}_i^m(\gamma/\sqrt{N}) > \widehat{r}_i^{m,*}(1/\sqrt{N})$ for any sufficiently large N. Since $\widehat{r}_i^m$ and $\widehat{r}_i^{m,*}$ are differentiable with $\widehat{r}_i^m(0) = \widehat{r}_i^{m,*}(0)$ (see Lemma 3 (c)), it depends only on the slopes of the functions $\widehat{r}_i^m$ and $\widehat{r}_i^{m,*}$ at zero whether $\widehat{r}_i^m(\gamma/\sqrt{N}) > \widehat{r}_i^{m,*}(1/\sqrt{N})$ for large N. Sufficient for the inequality to hold for any sufficiently large N is that $\gamma \cdot [\mathrm{d}/(\mathrm{d}x_i)\widehat{r}_i^m(x_i)]_{x_i=0} = \gamma \cdot [(1-\alpha) + \alpha h'(0)]$ strictly exceeds $[\mathrm{d}/(\mathrm{d}x_i)\widehat{r}_i^{m,*}(x_i)]_{x_i=0} = 1/N + (N-1)/N \cdot h'(0)$. Hence, the inequality holds for any sufficient large N if $\gamma \cdot [(1-\alpha) + \alpha h'(0)] > h'(0)$. Since this inequality is equivalent to $\gamma > \gamma^*$, I obtain Property 1.

I use the following definition in the remainder of the proof: For any fix $\gamma \in (\gamma^*, 1)$, let N_γ be some value such that the inequality in Property 1 holds for any $N > N_\gamma$.

Property 2: If $\gamma \in (\gamma^*, 1)$ and $N > N_\gamma$, then any element in $\widehat{r}_i^{m,*}([\gamma/\sqrt{N}, 1/\sqrt{N}])$ is strictly smaller than any element in $\widehat{r}_i^m([\gamma/\sqrt{N}, 1/\sqrt{N}])$. This is a direct consequence of Property 1 and monotonicity of $\widehat{r}_i^m$ and of $\widehat{r}_i^{m,*}$.

Property 3: If $\gamma \in (\gamma^*, 1)$ and $N > N_\gamma$, then $\max_{r' \in \widehat{r}_i^N([\gamma/\sqrt{N}, 1/\sqrt{N}])} -(r' - \widehat{r}_i^{m,*}(x_i))^2$ possesses for any $x_i \in [\gamma/\sqrt{N}, 1/\sqrt{N}]$ the same maximizer. Assume to the contrary that there exist $x_i', x_i'' \in [\gamma/\sqrt{N}, 1/\sqrt{N}]$ and $r', r'' \in \widehat{r}_i^N([\gamma/\sqrt{N}, 1/\sqrt{N}])$ with $r' < r''$ such that $-(r' - \widehat{r}_i^{m,*}(x_i'))^2 > -(r'' - \widehat{r}_i^{m,*}(x_i'))^2$ but $-(r'' - \widehat{r}_i^{m,*}(x_i''))^2 < -(r' - \widehat{r}_i^{m,*}(x_i''))^2$. These inequalities can be rewritten as $\widehat{r}_i^{m,*}(x_i') < (r' + r'')/2$ and $(r' + r'')/2 < \widehat{r}_i^{m,*}(x_i'')$. This implies $(r' + r'')/2 \in \widehat{r}_i^{m,*}([\gamma/\sqrt{N}, 1/\sqrt{N}])$. Analogously, that both reports r' and r'' are actually chosen by the agents for some attributes from $[\gamma/\sqrt{N}, 1/\sqrt{N}]$ requires $(r' + r'')/2 \in \widehat{r}_i^m([\gamma/\sqrt{N}, 1/\sqrt{N}])$. Since $\widehat{r}_i^{m,*}([\gamma/\sqrt{N}, 1/\sqrt{N}])$ and $\widehat{r}_i^m([\gamma/\sqrt{N}, 1/\sqrt{N}])$ do however not intersect by Property 2, it is not possible that $(r' + r'')/2$ falls simultaneously into both intervals. Contradiction.

Property 4: If $\gamma \in (\gamma^*, 1)$, then $\lim_{N\to\infty} Prob_X[\mathrm{m}(X) \in [\gamma/\sqrt{N}, 1/\sqrt{N}]] > 0$. This is a direct consequence from the asymptotic distribution of $\sqrt{N}\mathrm{m}(X)$ being normal (see, e.g., the Corollary to Theorem 13 in Ferguson (1996)).

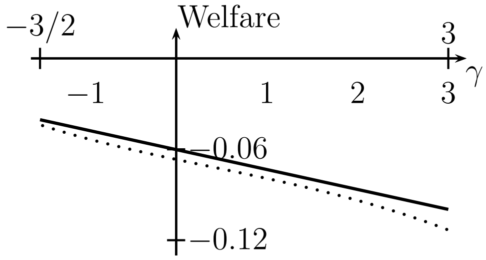
Property 5: If $\gamma \in (\gamma^*, 1)$, then $\lim_{N\to\infty} Var_X[\sqrt{N}\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X) \in [\gamma/\sqrt{N}, 1/\sqrt{N}]] > 0$. The asymptotic distribution of $\sqrt{N}\mathrm{m}(X)$ conditional on $\sqrt{N}\mathrm{m}(X) \in [\gamma, 1]]$ is a truncated normal distribution. In particular, it is non–degenerate. Since the slope of $\widehat{r}_i^{m,*}$ at zero converges towards $h'(0)$, $\sqrt{N}\widehat{r}_i^{m,*}(\mathrm{m}(X))$ is approximately $h'(0)\sqrt{N}\mathrm{m}(X)$ for large N. Since $h'(0) > 0$ by Lemma A1 (b) in Appendix A, the variance expression is bounded away from zero for large N.

Consider $\gamma \in (\gamma^*, 1)$ and $N > N_\gamma$. I have then

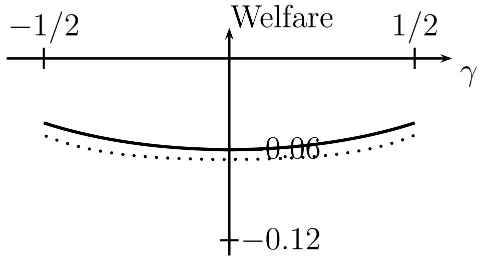
$$
\begin{aligned}
& N \cdot \mathbf{E}_X[-(\widehat{r}_i^N(\mathrm{m}(X)) - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2] \\
\leq\; & N \cdot \mathrm{Prob}_X[\mathrm{m}(X) \in [\gamma/\sqrt{N}, 1/\sqrt{N}]] \\
& \qquad \cdot \mathbf{E}_X[-(\widehat{r}_i^N(\mathrm{m}(X)) - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2|\mathrm{m}(X) \in [\gamma/\sqrt{N}, 1/\sqrt{N}]] \\
\leq\; & N \cdot \mathrm{Prob}_X[\mathrm{m}(X) \in [\gamma/\sqrt{N}, 1/\sqrt{N}]] \\
& \qquad \cdot \max_{r' \in \mathbb{R}} \mathbf{E}_X[-(r' - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2|\mathrm{m}(X) \in [\gamma/\sqrt{N}, 1/\sqrt{N}]] \\
=\; & -\mathrm{Prob}_X[\mathrm{m}(X) \in [\gamma/\sqrt{N}, 1/\sqrt{N}]] \\
& \qquad \cdot \mathrm{Var}_X[\sqrt{N}\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X) \in [\gamma/\sqrt{N}, 1/\sqrt{N}]]
\end{aligned}
$$

The transformations arise as follows: The first inequality follows from omitting the "loss" which arises when $\mathrm{m}(X) \notin [\gamma/\sqrt{N}, 1/\sqrt{N}]$. The second inequality follows from Property 3. The equality follows from using that the maximizer is by a reasoning like in the proof to Lemma 1 given by $r' = \mathbf{E}_X[\widehat{r}_i^{m,*}(\mathrm{m}(X))|\mathrm{m}(X) \in [\gamma/\sqrt{N}, 1/\sqrt{N}]]$ and from rewriting the expected value expression as a variance.

Since the right–hand side converges by Properties 4 and 5 to a strictly negative value as



(a) Quadratic densities



(b) Linear densities

Figure C.1: Lower bound on welfare of the optimal average mechanism (solid line) and upper bound on welfare of the optimal median mechanism (dotted line) [$N = 3$, $\alpha = 0$]

$N \to \infty$, $N \cdot \mathbf{E}_X[-(\widehat{r}_i^N(\mathrm{m}(X)) - \widehat{r}_i^{m,*}(\mathrm{m}(X)))^2]$ must be bounded away from zero for large N. This concludes the proof of Part a.

Part b: The optimal aggregation rule for $F \notin \mathcal{F}_s$. The result for distributions $F \notin \mathcal{F}_s$ is a direct consequence of Proposition 4 and Lemma 7 (b).

Part c: The asymptotically optimal mechanisms. By Part a and Part b, any asymptotically optimal mechanism relies on the average aggregation rule when $F \in \mathcal{F}_s \cap \mathcal{F}_h$ or $F \notin \mathcal{F}_s$. The asymptotic optimality of the binary average mechanism stated in the proposition is a direct consequence of Lemma 7 (a) and Lemma 4. That the same asymptotic welfare is approximated by the sequence of average mechanisms with interval report spaces stated in the proposition can be easily verified. q.e.d.

Appendix C. Details of the numerical result in Subsection 6.3

The upper bound on welfare attained by the optimal median mechanism $V^{*|\mathrm{m}(X)}$ depends only on the distribution F and the number of agents N. I can easily compute this bound numerically for any distribution F and any number of agents N. The welfare attained by any specific average mechanism $V(\phi_a, R)$ serves as a lower bound on the welfare attained by the optimal average mechanism. This lower bound depends on the selected report space R, the degree of interdependence α, the distribution F and the number of agents N. As the superiority of the average aggregation rule is least likely for $\alpha = 0$, it suffices to construct for any F and any N a report space R such that $V(\phi_a, R) > V^{*|\mathrm{m}(X)}$ for $\alpha = 0$. I construct now such report spaces for $N = 3$ and two specific classes of distributions with a normalized support of length 2.

Distributions with densities which are quadratic and symmetric around zero. Consider $\mathcal{X} = [-1, 1]$ and $f(x_i) = \gamma/2 \cdot x_i^2 + 1/2 - \gamma/6$ with parameter $\gamma \in (-3/2, 3)$. This class of distributions consists of symmetric distributions for which most probability mass lies either close to the median of the distribution (small γ) or close to the two endpoints of its support (large γ). The specific report space $R = [-1/2, 1/2]$ yields the result for the entire class of distributions. See Figure C.1a for an illustration of the upper bound on welfare attained by the optimal median mechanism (dotted curve) and the lower bound on welfare attained by the optimal average mechanism (solid curve) as a function of the parameter γ.

Distributions with linear densities. Consider $\mathcal{X} = [-2/3 \cdot \gamma - 1, -2/3 \cdot \gamma + 1]$ and $f(x_i) = \gamma x_i + 2/3 \cdot \gamma^2 + 1/2$ with $\gamma \in (-1/2, 1/2)$. The median of the distribution is $\eta = -(4\gamma^2 + 3 - 3\sqrt{4\gamma^2 + 1})/(6\gamma)$ and it differs from zero when $\gamma \neq 0$. An interval report space of length 1 which induces an unbiased reporting behavior yields the result. A report space with these properties is by Corollary 1 given by $R = [\widehat{r}_i^a(t', 0), \widehat{r}_i^a(t'+1, 0)]$ with t' being such that it solves $b^{a,I}(t', t'+1) = 0$. I obtain $R = [(3 - 4\gamma^2 - \sqrt{9 - 15\gamma^2})/(6\gamma) - 1/2, (3 - 4\gamma^2 - \sqrt{9 - 15\gamma^2})/(6\gamma) + 1/2]$. See Figure C.1b for an illustration. q.e.d.

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